Introduction:

Definition of a topological space A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:

1) $\phi, X \in \mathcal{T}$.

2) If $U_1, \ldots, U_n \in \mathcal{T}$ then so is $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

3) If $U_\alpha \in \mathcal{T}$ for all $\alpha \in \mathcal{J}$, for some index set $\mathcal{J}$, then $\bigcup_{\alpha \in \mathcal{J}} U_\alpha \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called open subsets of the topology $\mathcal{T}$. The pair $(X, \mathcal{T})$ is called a topological space.

A subset $C \subseteq X$ is called closed if $X \setminus C$ is open (i.e. $X \setminus C \in \mathcal{T}$).

Let $A \subseteq X$ be any subset. The closure of A, denoted $\bar{A}$ is the subset defined by

$$\bar{A} = \bigcap_{\substack{A \subseteq F \\ F \text{ closed}}} F \text{ , which is closed.}$$

Similarly, Interior of A, denoted $\operatorname{Int} A$ or $\mathring{A}$, is the subset defined by

$$\operatorname{Int} A = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U.$$

<u>Examples:</u> 1) Finite topologies

$X = \{a, b, c\}$, $\mathcal{T} = \{\phi, \{a,b,c\}, \{a\}, \{b\}, \{a,b\}\}$ is a topology on X.

2) X any set. The smallest topology on X is the topology $\{\phi, X\}$.
This topology is also called the weakest or coarsest topology on X.

3) X an set. The largest topology on X is the topology $\mathcal{P}(X)$.

This topology is also called the strongest or the finest topology on X.

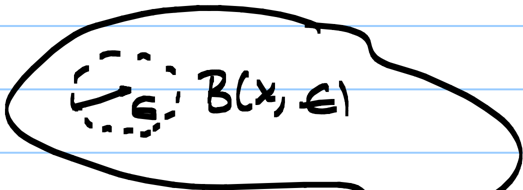
4) $X = \mathbb{R}$ $\mathcal{T}_{std} = \{U \mid U \subseteq \mathbb{R}$, if $x \in U$ then

$(x - \epsilon, x + \epsilon) \subseteq U$ for some $\epsilon > 0\}$.

$\mathbb{R}_{std}$, $\mathbb{R}^n_{std}$ $U \subseteq \mathbb{R}^n$ open if

for any $x \in U$ there is some $\epsilon > 0$ so that

$B(x, \epsilon) = \{y \in \mathbb{R}^n \mid \|x - y\| < \epsilon\} \subseteq U$.



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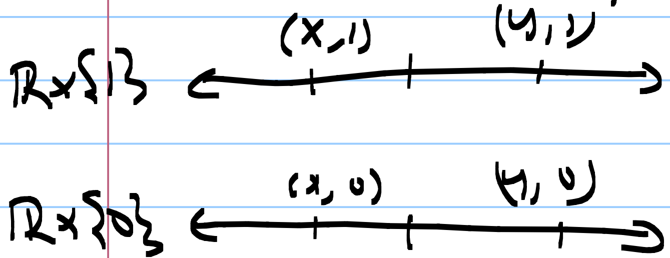
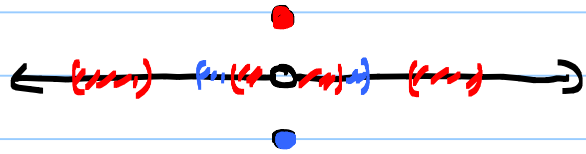
5) Real line with double origin.

$X = \mathbb{R} \times \{0,1\} \big/ (x,0) \sim (x,1)$ unless $x \neq 0$





<u>Definition:</u> A topological space $(X, \mathcal{T})$ is called T_0 if for any $x, y \in X$ with $x \neq y$ then there is an open set U so that either

$(x \in U$ and $y \notin U)$ or $(y \in U$ and $x \notin U)$.

Similarly, X is called T_1 if for any $x, y \in X$ with $x \neq y$ there is an open set U with

$x \in U$ and $y \notin U$.

Finally, X is called T_2 (or Hausdorff) if for any $x, y \in X$ with $x \neq y$ there are open subsets U, V so that

$x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Clearly, $T_2 \Rightarrow T_1 \Rightarrow T_0$.

<u>Example</u> The real line with double origin is is T_1 but <u>not</u> T_2.

Equivalence of Topologies:

A function $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called continuous if

$f^{-1}(U)$ is open $(f^{-1}(U)\in\mathcal{T}_X)$

U is open in Y $(U\in\mathcal{T}_Y)$.

A continuous bijection $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is called a homeomorphism if its inverse

$f^{-1}:(Y,\mathcal{T}_Y)\to(X,\mathcal{T}_X)$ is also continuous.

In this case $(X,\mathcal{T}_X)$ and $(Y,\mathcal{T}_Y)$ are called homeomorphic. We regard homeomorphic spaces as the same in topology.

Basis and Subbasis:

Let $(X,\mathcal{T})$ be a space. A subcollection $\mathcal{B}$ of open subsets of $(X,\mathcal{T})$ is called basis if the followings are satisfied.

1) For any open subset U of X and point $x\in U$ there is some $B\in\mathcal{B}$ so that

$$x\in B\subseteq U.$$

2) If $B_1,B_2\in\mathcal{B}$ and $x\in B_1\cap B_2$ then there is $B_3\in\mathcal{B}$ so that

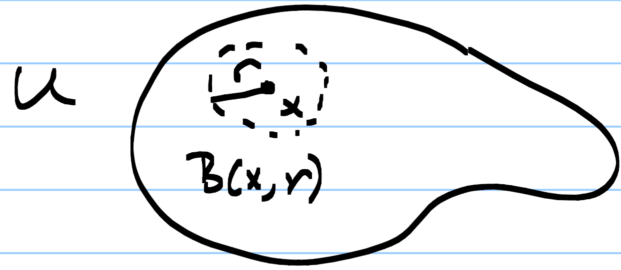
$$x\in B_3\subseteq B_1\cap B_2.$$

Video 2

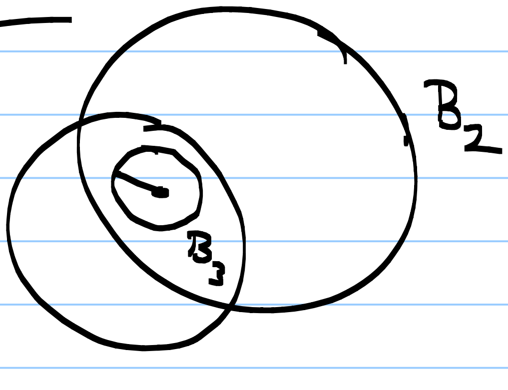
<u>Example:</u> $X = \mathbb{R}^2_{std}$

$\mathcal{B} = \{B(x,r) \mid x \in \mathbb{R}^2, r > 0\}$

1)



2)

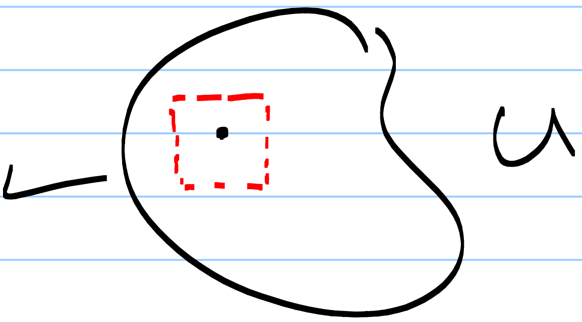


Hence, $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}^2$.

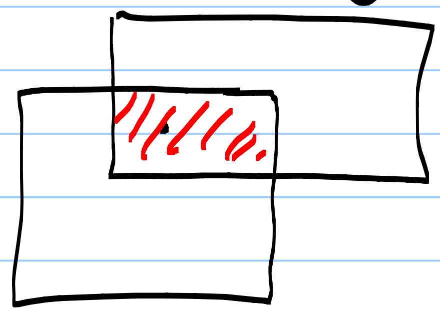
2) $\mathcal{C} = \{(a,b) \times (c,d) \mid a<b, \ c<d \in \mathbb{R}\}$

is also a basis for the standard topology on $\mathbb{R}^2$.

1)



2)



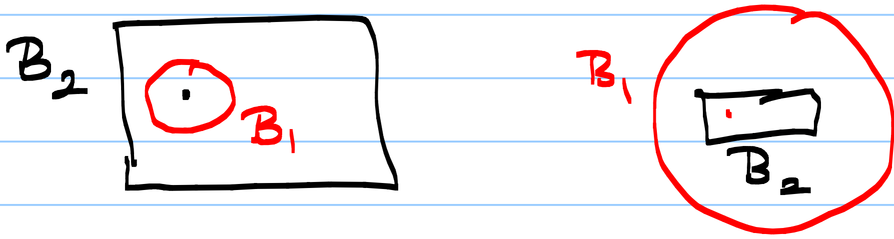
<u>Comparison of Topologies:</u> Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. We say that $\mathcal{T}_1$ is stronger or finer than $\mathcal{T}_2$ if

$$\mathcal{T}_2 \subseteq \mathcal{T}_1.$$

Remark: Let $\mathcal{B}_i$ be bases for $\mathcal{T}_i$, $i=1,2$, two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X.

Then $\mathcal{T}_1$ is stronger that $\mathcal{T}_2$ if and only if the following holds:

For any $B_2 \in \mathcal{B}_2$ and point $x \in B_2$ then is some $B_1 \in \mathcal{B}_1$ so that $x \in B_1 \subseteq B_2$.



Definition: Let X be any set and $\mathcal{B}$ be a collection of subsets of X. The collection of subsets consisting of finite intersections of elements of $\mathcal{B}$ and arbitrary unions of finite intersections form a topology on X and it is called the topology generated by subbasis $\mathcal{B}$.

$$X = \bigcup_{B \in \mathcal{B}} B$$

New Topologies from Old Ones

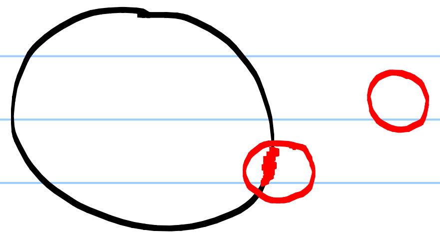
Subspace Topology $(X, \mathcal{T})$ topological space.

Any subset A of X inherits a topology from

$(X, \mathcal{T})$ called the subspace topology, defined by

$$\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}.$$

<u>Example:</u> $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \subseteq \mathbb{R}^2_{std}$



<u>Definition:</u> Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a one and continuous map. Then $f(X)$, the image of X, is a subspace of Y.

If the map $f : (X, \mathcal{T}_X) \to (f(X), \mathcal{T}_Y|_{f(X)})$ is a homeomorphism (i.e., "f^{-1}" is also continuous, then f is called a topological embedding of X into Y.

<u>Example</u> $X = [0, \infty)$ with the basis

$$\mathcal{B} = \{ (a,b) \mid 0 < a < b \} \cup \{ [0,a) \cup (b, \infty) \mid a, b > 0 \}.$$

One can check that $\mathcal{B}$ is a basis for a topology on X.

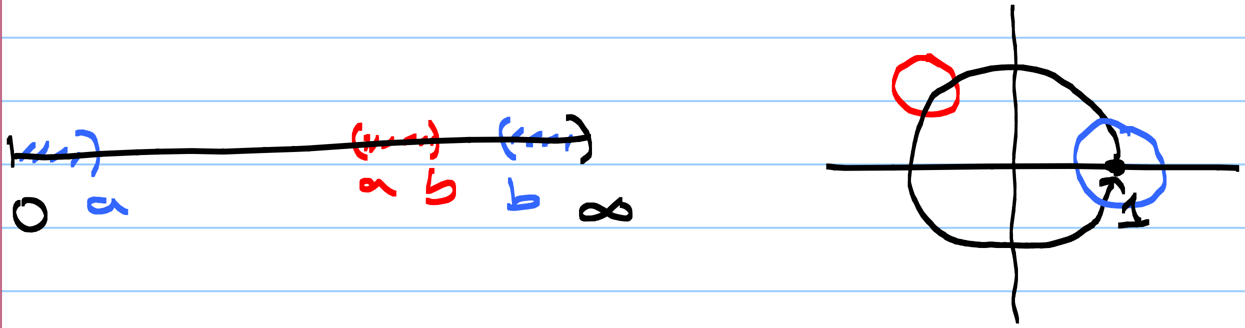
<u>Claim:</u> The topological space $(X, \mathcal{T})$, where $\mathcal{T}$ is generated by the basis $\mathcal{B}$ is homeomorphic to the subspace topology on S^1.

Video 3

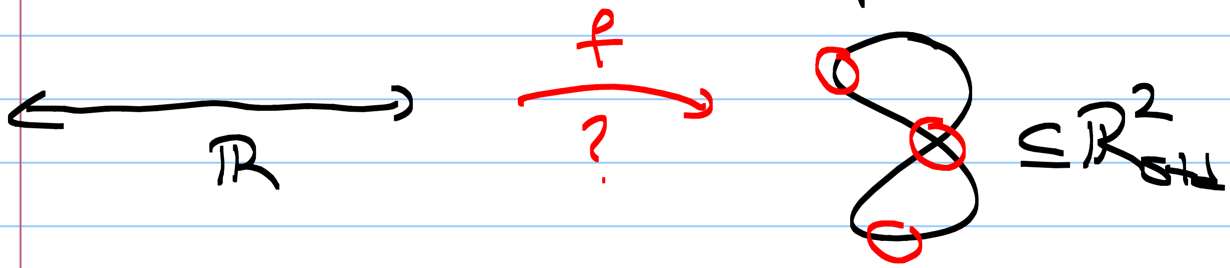
A homeomorphism is given by

$f:[0,\infty) \longrightarrow S^1$, $f(t) = e^{2\pi i t/1+t}$

$f(t) = \left(\cos \frac{2\pi t}{1+t}, \sin \frac{2\pi t}{1+t}\right)$, $t \in [0,\infty)$.



Exercise: Put a topology on $\mathbb{R}$ so that it is homeomorphic to the figure eight in $\mathbb{R}^2_{std}$.



Product Spaces

$(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ a family of topological spaces

$\prod_{\alpha\in\Lambda} X_\alpha = \{ f:\Lambda \to \dot\bigcup_{\alpha\in\Lambda} X_\alpha \mid f(\alpha)\in X_\alpha, \forall \alpha\in\Lambda\}$.

$\prod X_\alpha \neq \emptyset$ by the Axiom of Choice

Product topology on $\prod X_\alpha$ is given by the basis $\mathcal{B}$ consisting of elements of the form $\prod_{\alpha\in\Lambda} U_\alpha$, $U_\alpha \subseteq X_\alpha$ open and

$U_\alpha = X_\alpha$ for all but finitely many $\alpha\in\Lambda$.

Box Topology: This is given by the basis consisting of products of the form

$\prod_{\alpha\in\Lambda} U_\alpha$, $U_\alpha \subseteq X_\alpha$ open for all α.

Box topology is much stronger than the product topology.

Fact: $\varphi : X \to \prod_\alpha X_\alpha$, X, X_α top. spaces.

$$P_\beta : \prod_\alpha X_\alpha \longrightarrow X_\beta, \qquad f\in\prod_\alpha X_\alpha, f:\Lambda\to\dot{\bigcup_\alpha} X_\alpha$$
$$f \longmapsto f(\beta)$$

Each P_β is clearly continuous.

φ is continuous if and only if each coordinate function

$$P_\beta\circ\varphi : X \longrightarrow X_\beta \quad \text{is continuous.}$$

Quotient Topology

Let $f: X \to Y$ be an onto map so that X is a topological space and Y is any set. Call a subset V of Y open if and only if $f^{-1}(V)$ is open in X.

This defines a topology on Y and it is the strongest topology on Y so that f is continuous. This topology is called the quotient topology on Y determined by $f: X \to Y$.

Remark: Any onto map $f: X \to Y$ defines an equivalence relation on X as follows:

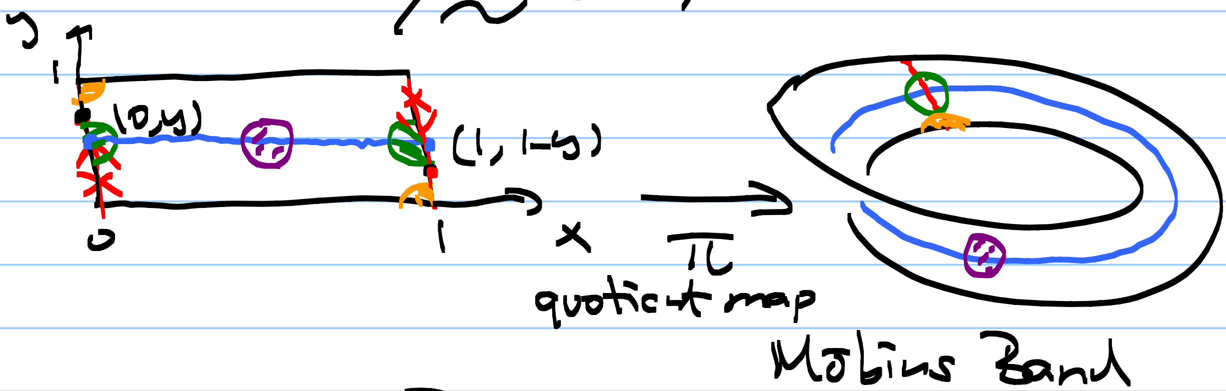
$$x_1, x_2 \in X, \quad x_1 \sim x_2 \iff f(x_1) = f(x_2).$$

$[x] = \{x' \in X \mid x' \sim x\}$ the equivalence class of x.

The set equivalence classes can be identified with Y.

$$X/\sim \; = Y$$

Ex:



$$MB = [0,1] \times [0,1] /\sim$$

$(x_1, y_1) \sim (x_2, y_2)$ if and only if $|x_1 - x_2| = 1$ and $y_1 + y_2 = 1$.

Proposition: Let $\pi : X \to Y$ be a quotient space. Let $f : Y \to Z$ be a map. Then f is continuous if and only if $f \circ \pi : X \to Z$ is continuous.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Corollary: Let $\pi : X \to Y$ be a quotient map and $g : X \to Z$ be a map.

$$\begin{tikzcd} x \arrow[d, mapsto] & X \arrow[r, "g"] \arrow[d, "\pi"'] & Z \\ {[x]} & Y \arrow[ur, dashed, "?f"'] & \end{tikzcd}$$

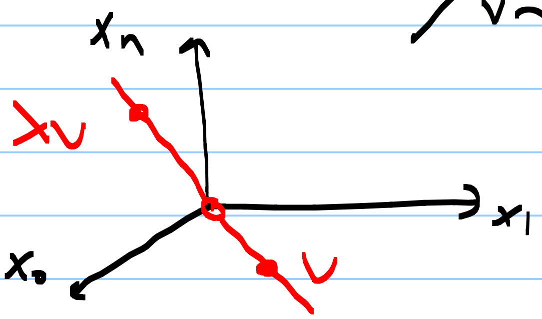
There is a map $f : Y \to Z$ such that $f \circ \pi = g$ if and only if g is constant on each $\pi^{-1}(y)$ (fiber).

Moreover, if such f exists then f is continuous if and only if g is continuous.

In this case, we say that g descends to the quotient space Y.

Example: $\mathbb{RP}^n$: Real Projective Space.

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} \Big/ v \sim \lambda v, \quad v = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}, \quad \lambda \in \mathbb{R} \setminus \{0\}$$



$\mathbb{RP}^n$: the set of all lines in $\mathbb{R}^{n+1}$ containing the origin.

Consider the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$

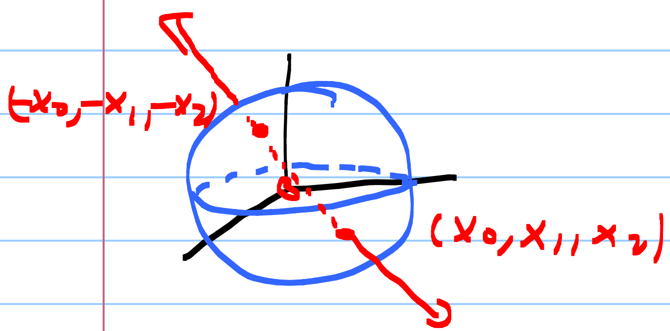
$v = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, $\pi(v) \doteq [x_0 : x_1 : \dots : x_n]$

$[x_0 : \dots : x_n] = \{\lambda (x_0, \dots, x_n) \mid \lambda \in \mathbb{R} \setminus \{0\}\}$.

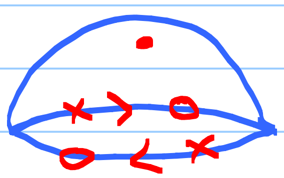
We endow $\mathbb{RP}^n$ with the quotient topology induced by the map π, where $\mathbb{R}^{n+1} \setminus \{0\}$ has the Euclidean (subspace) topology.

Example: $\mathbb{RP}^2$ the Real Projective Geometry

$$\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\} \big/ v \sim \lambda v, \quad v = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}, \ \lambda \in \mathbb{R} \setminus \{0\}$$



$$\mathbb{RP}^2 = S^2 \big/ (x_0, x_1, x_2) \sim (-x_0, -x_1, -x_2)$$



$\mathbb{RP}^2$ does not embed into $\mathbb{R}^3$.
(Not easy to prove!)

Proposition: $\mathbb{RP}^2$ embeds into $\mathbb{R}^5$.

Proof:
$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^5 \\ \mathbb{RP}^2 \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

want: g a homeomorphism onto its image.

$\pi(x_0, x_1, x_2) = [x_0 : x_1 : x_2] = \pi(-x_0, -x_1, x_2).$

$f(x_0, x_1, x_2) = (x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2)$

$(x_0^2 + x_1^2 + x_2^2 = 1)$ f is clearly continuous on S^2

$$\begin{tikzcd} S^2 \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{R}^5 \\ \mathbb{RP}^2 \arrow[ur, "g"', "\exists !"] & \end{tikzcd}$$

$f = g \circ \pi$

g is also continuous since f is.

Since S^2 is compact so is its image $\pi(S^2)$. Hence, $\mathbb{RP}^2$ is a compact topological space.

Since $\mathbb{RP}^2$ is compact and $g : \mathbb{RP}^2 \to \mathbb{R}^5$ is continuous g is a homeomorphism onto its image.

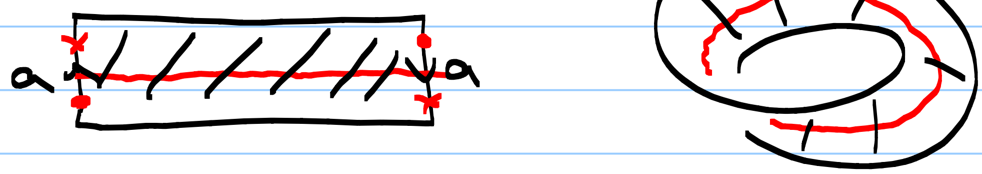
Here, we use the fact that f is 2:1 and thus g is one to one.

$f(x_0, x_1, x_2) = f(y_0, y_1, y_2) \iff$

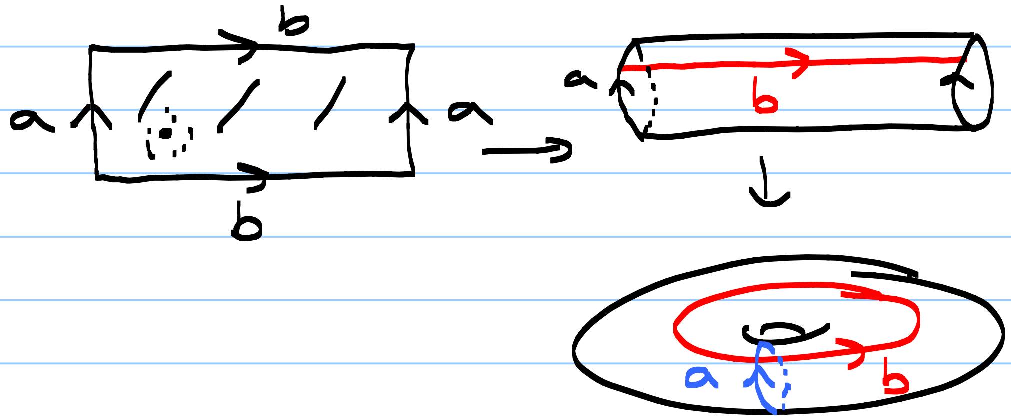
$(x_0, x_1, x_2) = \pm(y_0, y_1, y_2)$

More Examples with Quotient topology

1) Möbius Band



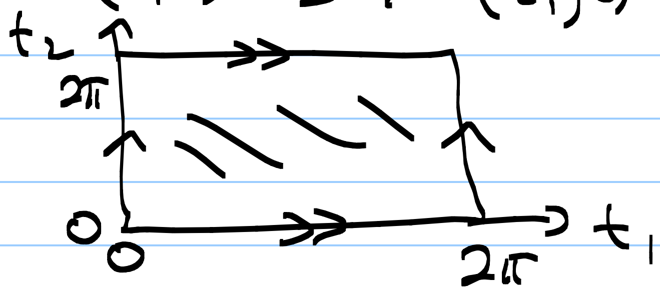
2) Torus T^2



T^2 is homeomorphic to $S^1 \times S^1$ and it embeds in $\mathbb{R}^3$.

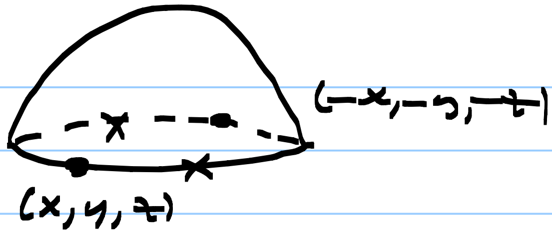
$$\begin{tikzcd}
\varphi : [0,2\pi]\times[0,2\pi] \arrow[r] \arrow[d] & S^1\times S^1 \\
(t_1, t_2) \arrow[r, mapsto] & (\cos t_1, \sin t_1, \cos t_2, \sin t_2) \\
{[0,2\pi]\times[0,2\pi] \big/ \begin{smallmatrix}(0,t_2)\sim(2\pi,t_2)\\ (t_1,0)\sim(t_1,2\pi)\end{smallmatrix}} \arrow[ur, "\phi\ \exists!"] &
\end{tikzcd}$$

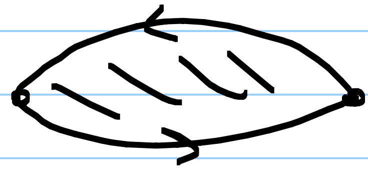
$[0,2\pi]\times[0,2\pi] \,/\, (0,t_2)\sim(2\pi,t_2)$
$(t_1 \;,\; t_2) \qquad\;\; (t_1,0)\sim(t_1,2\pi)$, for all t_1, t_2.



ϕ is a homeomorphism.

Video 5

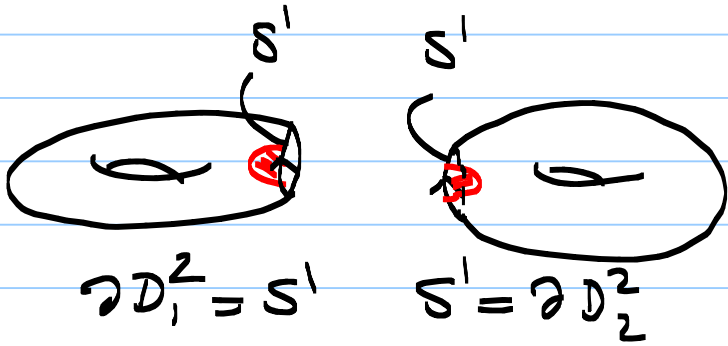
3) $\mathbb{RP}^2 =$ 

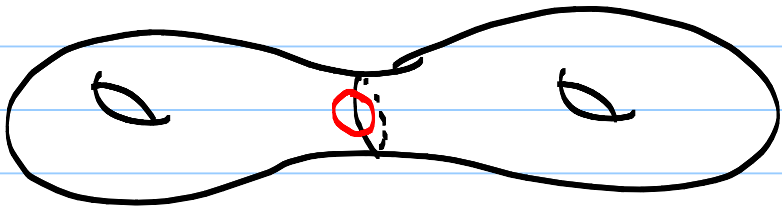
$=$ 

4) Connected Sum (of Surfaces):

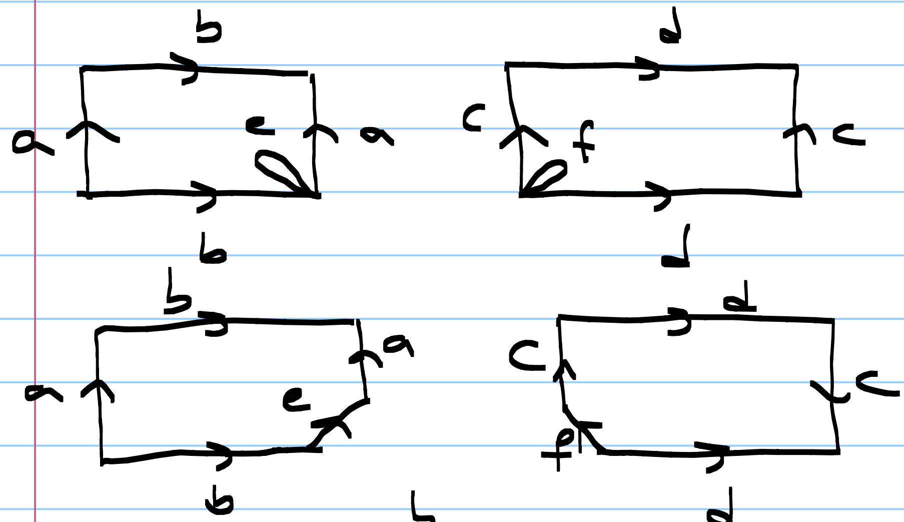
$$T^2 \# T^2 = (T^2 \setminus D_1^2) \sqcup (T^2 \setminus D_2^2) \,/\, x \sim f(x)$$

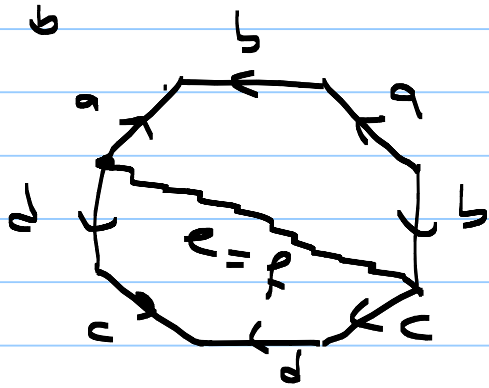
$f: \partial D_1^2 \to \partial D_2^2$
a homeomorphism



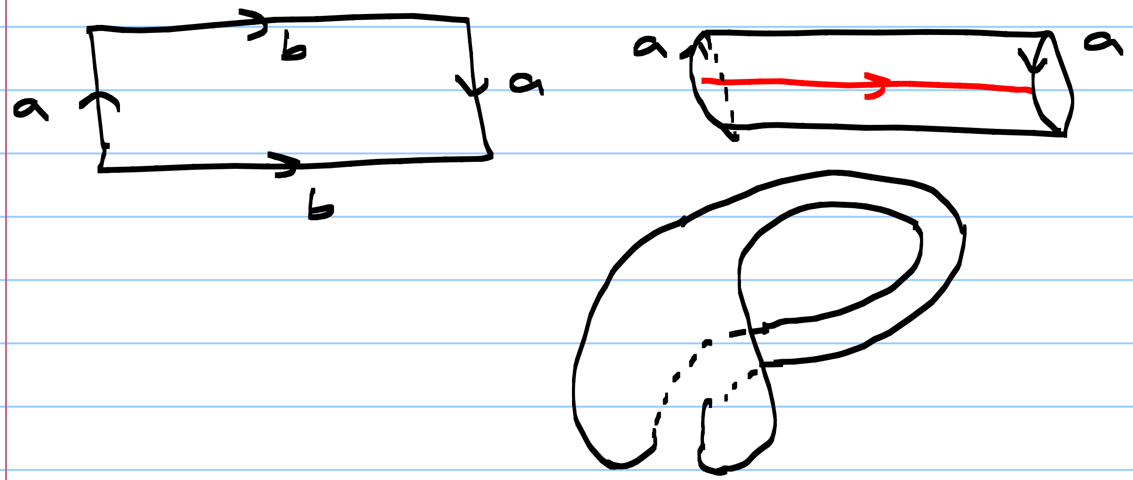


Genus two orientable surface





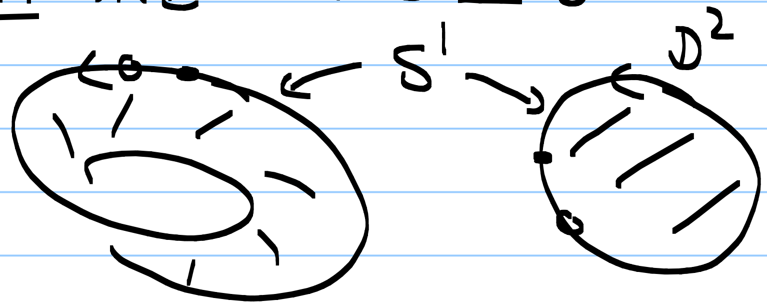
Exercise: Klein Bottle



Prove that $KB = \mathbb{RP}^2 \# \mathbb{RP}^2$ and

$$KB \# \mathbb{RP}^2 = \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = T^2 \# \mathbb{RP}^2.$$

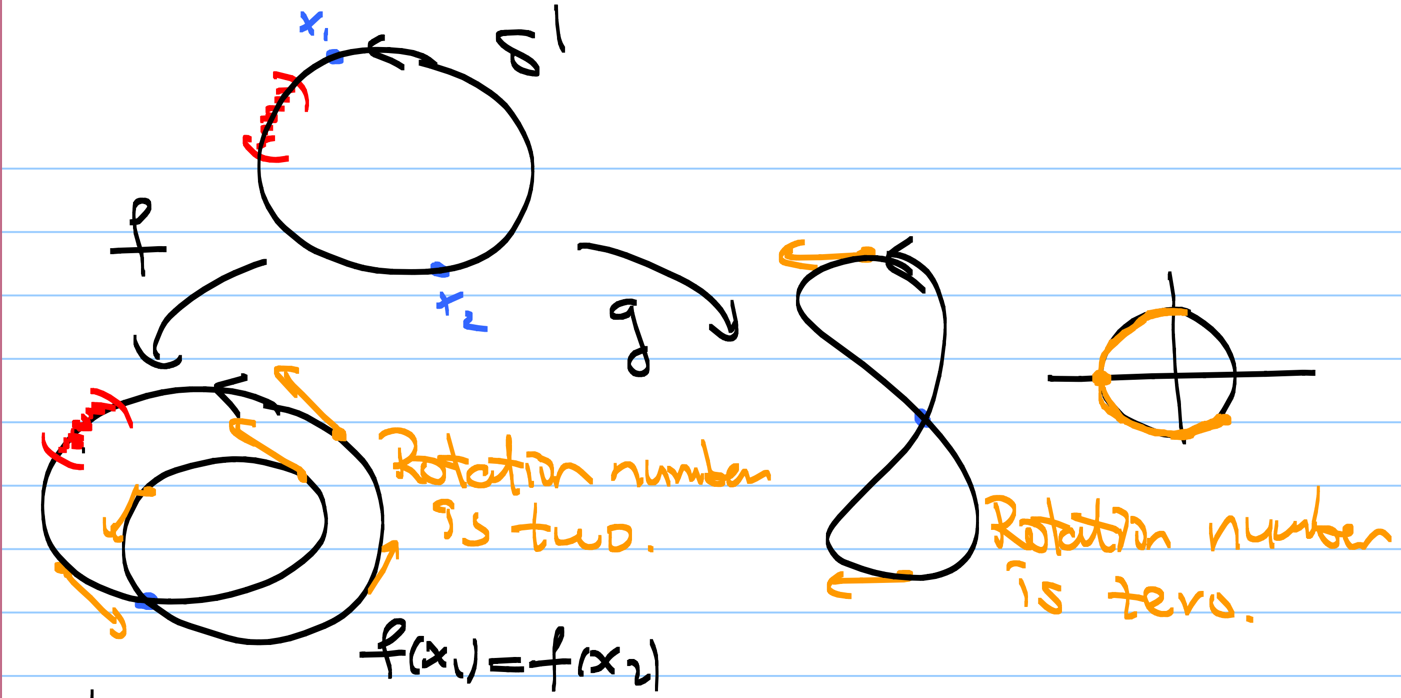
Hint: $\mathbb{RP}^2 = MB \amalg D^2$



Question: Why algebraic topology?

Answer: Comparing topological spaces is generally much more difficult than comparing algebraic objects?

Example: How can we distinguish the two immersions of the S^1 into $\mathbb{R}^2$ given below.



<u>not</u> an embedding.

The answer to this question is No!
The above immersions are <u>not</u> homotopic through immersions.

Rotation number is invariant under homotopies (through immersions) and thus the above immersions are <u>not</u> homotopic.

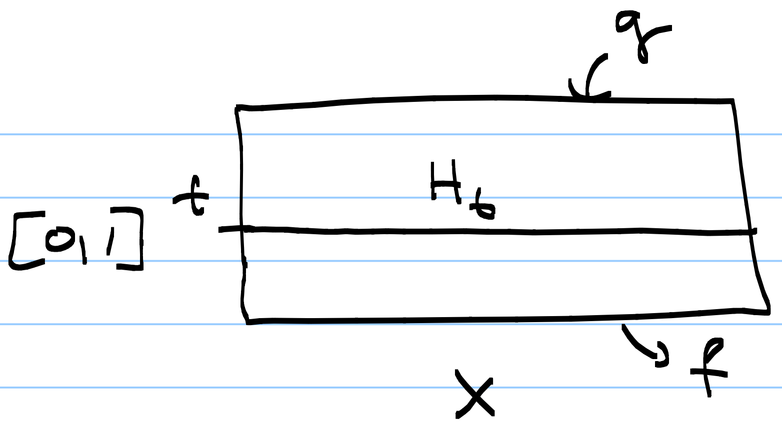
<u>Definition of Homotopy:</u>

Let $f: X \to Y$ and $g: X \to Y$ be two continuous maps of topological spaces. We say that f and g are homotopic maps if there is a continuous map

$$H: X \times I \to Y, \quad I = [0,1], \text{ so that}$$

$$H(x,0) = f(x) \text{ and } H(x,1) = g(x), \text{ for all } x \in X.$$

Video 6



$H_t = H(x,t)$, $H_0 = f$, $H_1 = g$

(Winding and Rotation numbers, Math 709, Video 22)

Definition (Relative Homotopy)

Let (X,A) pair of topological spaces ($A \subseteq X$ subspace) and Y any topological space.

Let $f: X \longrightarrow Y$ and $g: X \longrightarrow Y$ be continuous maps. We say that f and g are homotopic relative to A if there is a homotopy

$H: X \times I \longrightarrow Y$ such that

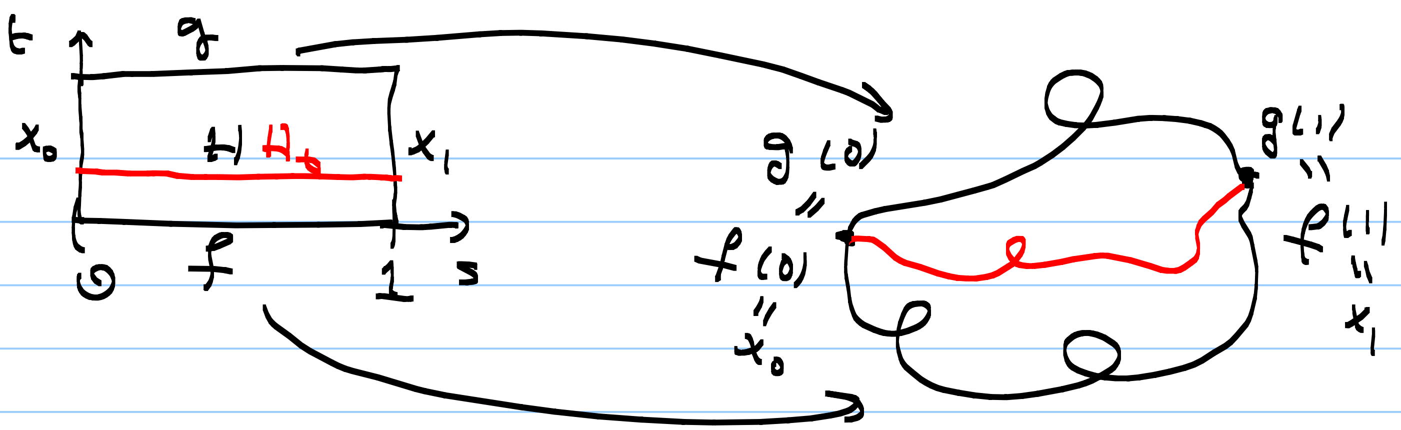
$H(x,0) = f(x)$, $H(x,1) = g(x)$, for all $x \in X$ and

$H(x,t) = f(x) = g(x)$ for all $x \in A$, and for all $t \in [0,1]$.

Example: $X = [0,1]$, $A = \partial X = \{0,1\}$.

$f, g: [0,1] \longrightarrow Y = \mathbb{R}^2$ continuous maps so that

$f(0) = g(0)$ and $f(1) = g(1)$.



f is homotopic to g rel $\{0,1\}$.

__Homotopy Equivalence__ Let $f: X \longrightarrow Y$ and $g: Y \to X$ be two continuous functions. If the compositions

$$f \circ g : Y \longrightarrow Y \text{ and } g \circ f : X \longrightarrow X \text{ are}$$

homotopic to $id_Y : Y \longrightarrow Y$ and $id_X : X \to X$, respectively, then we say that the spaces X and Y are homotopy equivalent.

__Example:__ $X = \mathbb{R}^n$, $Y = \{0\}$, $0 \in \mathbb{R}^n$, the origin.

$f: \mathbb{R}^n \longrightarrow \{0\}$, $f(x) = 0$, $\forall x \in \mathbb{R}^n$,

$g: \{0\} \longrightarrow \mathbb{R}^n$, $g(0) = 0$.

$f \circ g : \{0\} \longrightarrow \{0\}$, $f \circ g = id_{\{0\}}$

$H_1 : \{0\} \times I \longrightarrow \{0\}$, $H_1(\,,t) = f \circ g$

$g \circ f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(g \circ f)(x) = 0$, for all $x \in \mathbb{R}^n$.

Consider the homotopy

$H_2 : \mathbb{R}^n \times I \to \mathbb{R}^n$, $H_2(x,t) = tx$, when

$H_2(x,0) = 0.x = 0 = (g\circ f)(x)$, for all $x \in \mathbb{R}^n$,

and $H_2(x,1) = 1.x = x = Id_{\mathbb{R}^n}(x)$, for all $x \in \mathbb{R}^n$.

Hence the spaces $\mathbb{R}^n$ and $\{0\}$ are homotopy equivalent.

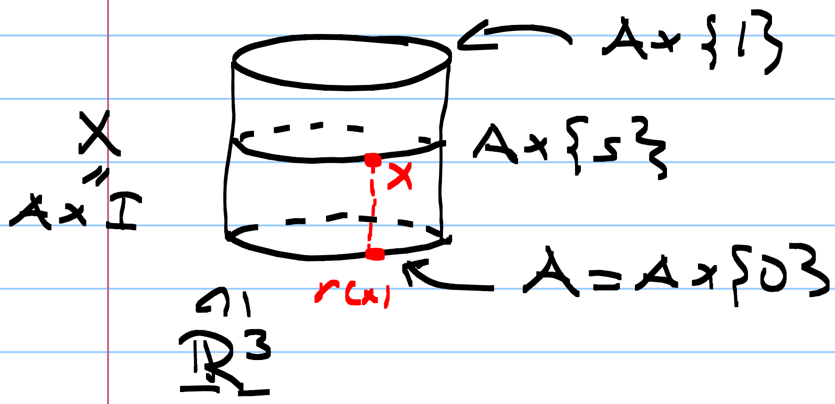
We'll see that Homotopy Equivalence is an equivalence relation on the collection of topological spaces.

<u>Definition:</u> Let (X, A) be a topological pair.

A continuous map $r: X \to A$ is called a retraction if $r(a) = a$ for all $a \in A$.

A retraction $r: X \to A$ is called deformation retraction if there is a homotopy

$H: X \times I \to X$ so that $H(x,0) = x$ for all $x \in X$ and $H(x,1) = r(x)$.



$H(x,t) = (1-t)x + t\,r(x)$,
$H(x,0) = x$, for all $x \in X$.
$H(x,1) = r(x)$, for all $x \in X$.

Exercise: A deformation retraction is a homotopy equivalence.

Definition: A topological space is called contractible if the identity map $id_X : X \to X$ is homotopic to a constant map

$$f : X \to X, \quad f(x) = x_0, \text{ for some } x_0 \in X \text{ and for all } x \in X.$$

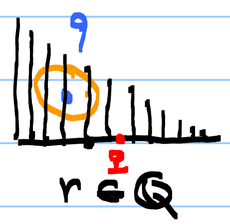
Example: The homotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ by

$H(x,t) = tx$ gives a contraction of $\mathbb{R}^n$ to the point $\{0\}$.

$H(x,1) = x = Id_{\mathbb{R}^n}(x)$, $\forall x \in \mathbb{R}^n$, and

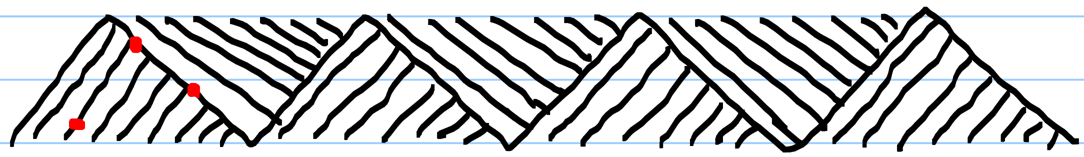
$H(x,0) = 0$, $\forall x \in \mathbb{R}^n$.

Remark: Exercise #6 of Chapter 0.

a) 

$X = \{(x,y) \mid x \in [0,1], y = 0\}$
$\cup \{(x,y) \mid x \in [0,1] \cap \mathbb{Q}, y \in [0, 1-x]\}$

X deformation retracts to any point on the x-axis but <u>not</u> to any other point.

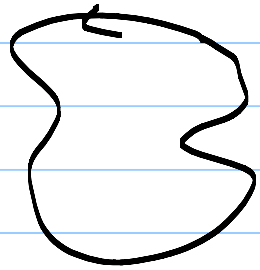
c) Y 

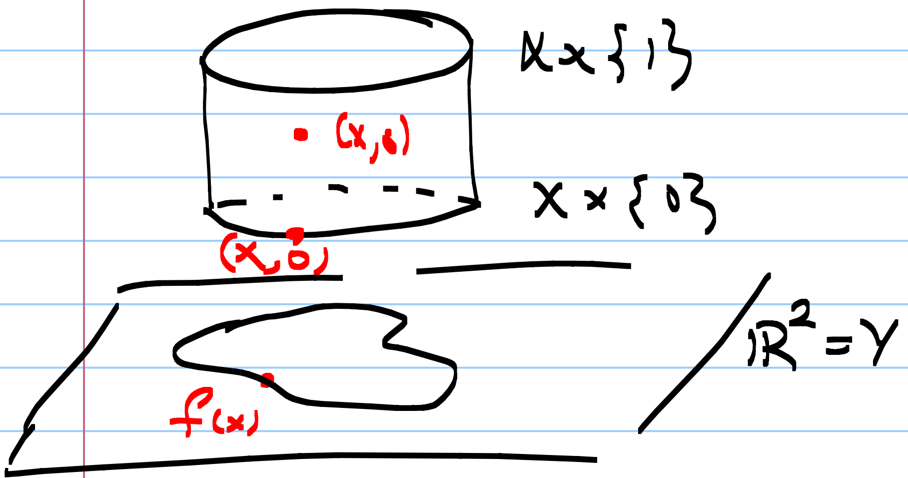
Y is contractible but does not deformation retract to any point.

Mapping Cylinder: Let $f: X \to Y$ be any continuous map. The topological space

$$M_f \doteq X \times I \,\dot\cup\, Y \Big/ (x,0) \sim f(x),\ \forall x \in X$$

$X = S^1$, $Y = \mathbb{R}^2$, $f: S^1 \to \mathbb{R}^2$





Proposition: M_f deformation retracts onto Y.

Proof: $M_f = X \times \underset{s}{I} \,\dot\cup\, Y \Big/ (x,0) \sim f(x),\ \forall x \in X.$

$H: M_f \times \overset{t}{I} \longrightarrow M_f$, $H((x,s),0) = (x,s)$ for all $(x,s) \in M_f$, and

$H((x,s),1) = f(x)$, for all $(x,s) \in M_f$.

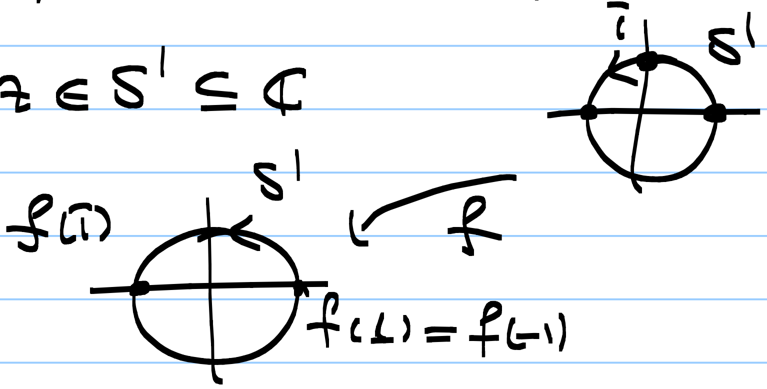
$r: M_f \longrightarrow Y$, $r(x,s) = f(x)$

Also define H on Y as identity. Then H is continuous and gives the desired deformation retraction.

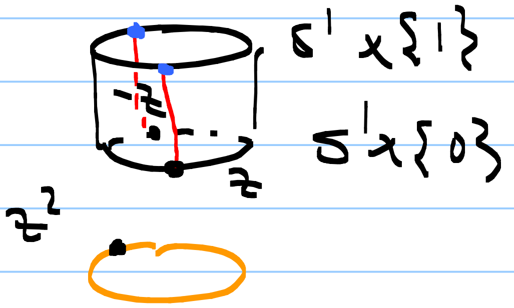
Example: Let $f: S^1 \to S^1$ be defined as

$f(z) = z^2$, $z \in S^1 \subseteq \mathbb{C}$

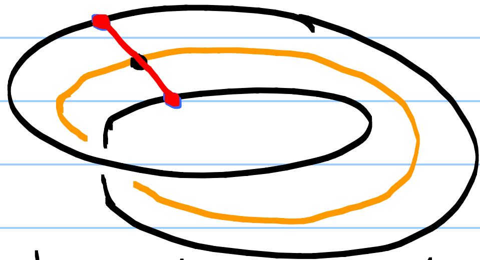
S^1 the unit circle in $\mathbb{C}$.



$X = S^1 = Y$, $M_f = S^1 \times I \cup S^1 / (z,0) \sim f(z) = z^2$



$M_f = MB$



For a proof just cut the above Möbius Band the the orange center circle.

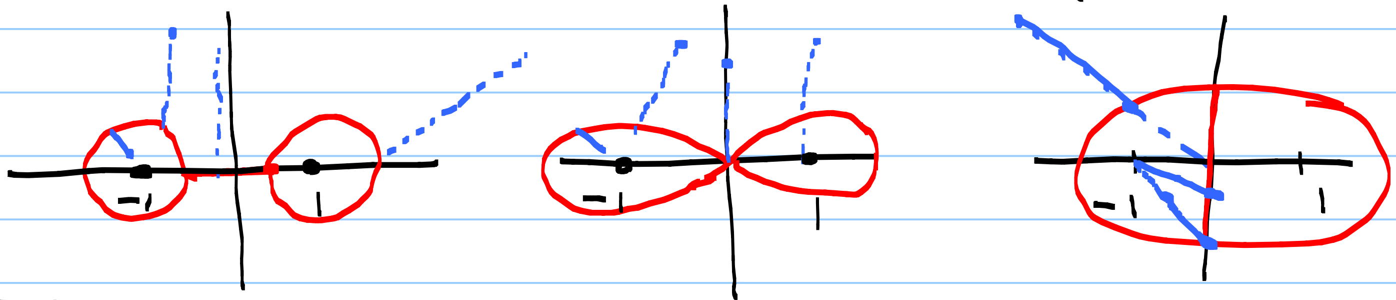
Proposition: Homotopy equivalence is an equivalence relation.

Remark: Deformation retraction is not an equivalence relation.

Remark: Deformation retraction is not an equivalence relation.

Video 8

Example: The three subspaces of $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ below are deformation retraction of $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ but they are <u>not</u> deformation retraction of each other:

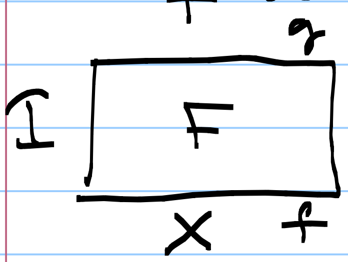


They are not deformation of each other because none of them is a subspace of any other.

Proposition: Homotopy equivalence is an equivalence relation.

Fact 1) If $f: X \to Y$ is homotopic to some $g: X \to Y$ then g is homotopic to f.

Proof Let $F: X \times I \to Y$ be a homotopy from f to g.



$F(x,0) = f(x)$, $F(x,1) = g(x)$, for all $x \in X$.

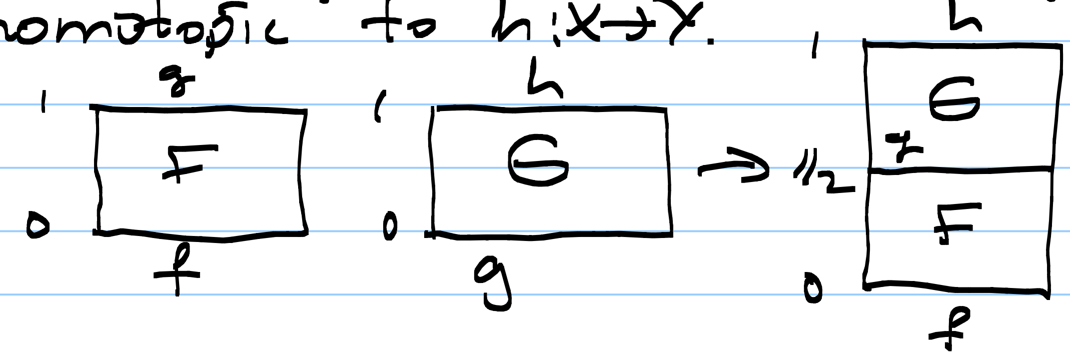
Note that $G: X \times I \to Y$, $G(x,t) = F(x, 1-t)$ is a homotopy from

$$G(x,0) = F(x, 1-0) = F(x,1) = g(x) \text{ to}$$
$$G(x,1) = F(x, 1-1) = F(x,0) = f(x), \text{ for all } x \in X.$$

Fact: Clearly any function $f:X\to Y$ is homotopic to itself via the homotopy $F(x,t)=f(x)$, $x\in X$, $t\in[0,1]$.

Fact: If $f:X\to Y$ is homotopic to $g:X\to Y$ and g is homotopic to $h:X\to Y$, then f is homotopic to $h:X\to Y$.

Proof



Let $F:X\times I\to Y$ and $G:X\times I\to Y$ be homotopies from f to g and g to h, respectively. Then

$$H:X\times I\to Y \text{ by } H(x,t)=\begin{cases}F(x,2t), & 0\le t\le 1/2\\ G(x,2t-1), & 1/2\le t\le 1\end{cases}$$

is the desired homotopy (pasting lemma).

So these three facts show that being homotopic is an equivalence relation.

Proof of the proposition:

1) Reflexivity ✓

2) Symmetry: Assume that X and Y are homotopy equivalent. Then there are maps $f:X\to Y$ and $g:Y\to X$ such that

$f \circ g : Y \to Y$ and $g \circ f : X \to X$ are homotopic to id_Y and id_X, respectively. This is clearly symmetric.

3) Assume X is homotopy equivalent to Y and Y is homotopy equivalent to Z.

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] & Y \arrow[l, bend left, "g_1"] \arrow[r, bend left, "f_2"] & Z \arrow[l, bend left, "g_2"] \end{tikzcd}$$

$g_1 \circ f_1 \sim id_X$, $f_1 \circ g_1 \sim id_Y$, $g_2 \circ f_2 \sim id_Y$ and $f_2 \circ g_2 \sim id_Z$.

<u>must show:</u> $g_1 \circ g_2 \circ f_2 \circ f_1 \sim id_X$ and

$f_2 \circ f_1 \circ g_1 \circ g_2 \sim id_Z$

$$g_1 \circ \underbrace{g_2 \circ f_2}_{} \circ f_1 \sim g_1 \circ id_Y \circ f_1 = g_1 \circ f_1 \sim id_X$$

$t = 0 \to t_1 = 1/2$ $\qquad$ $t = 1/2,\ t_1 = 1$

Let $\varphi : Y \times I \to Z$ be a homotopy from $g_2 \circ f_2$ to id_Y

Define $\tilde{\varphi} : X \times I \to X$, $\tilde{\varphi}(x,t) = g_1 \circ \varphi \circ f_1$

Video 9

$$\tilde{\varphi}(x,t) = g_1(\varphi(f_1(x),t))$$

$$\tilde{\varphi}(x,0) = g_1(\varphi(f_1(x),0)) = g_1(g_2\circ f_2)(f_1(x))$$
$$= (g_1\circ g_2\circ f_2\circ f_1)(x)$$

$$\tilde{\varphi}(x,1) = g_1(\varphi(f_1(x),1)) = g_1(f_1(x)) = (g_1\circ f_1)(x)$$

$\tilde{\varphi}(x,1)$ is not id_X but it is homotopic to id_X. Composing $\tilde{\varphi}$ with a homotopy taking $g_1\circ f_1$ to id_X we see that

$g_1\circ g_2\circ f_2\circ f_1$ is homotopic to id_X.

This finishes the proof of the proposition. ▪

Corollary 0.21 (From the Book)

Two spaces X and Y are homotopy equivalent if and only if there is a third space Z which deformation retracts onto X and Y.

Cell Complexes: We'll build a space inductively as follows:

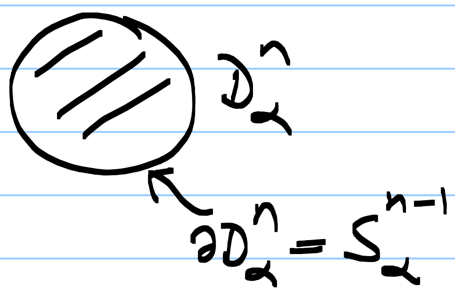
1) Start with a discrete set of points X^0, whose elements are called 0-cells.

2) Form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$\varphi_\alpha : S_\alpha^{n-1} = \partial D_\alpha^n \longrightarrow X^{n-1}.$

Hence, $X^n = X^{n-1} \coprod_\alpha D_\alpha^n \big/ x \sim \varphi_\alpha(x),\ x \in \partial D_\alpha^n,$

where e_α^n is $\mathrm{Int}(D_\alpha^n)$.



3) One can stop at some stage n and let $X = X^n$ or continue indefinitely, setting $X = \bigcup_n X^n$. In this case, a subset A of X will be called open (closed) if and only if $A \cap X^n$ is open (resp. closed) in X^n, for all n.

Such a space is called a CW-complex.

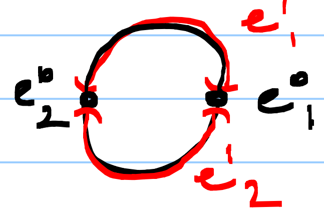
C : closure finite

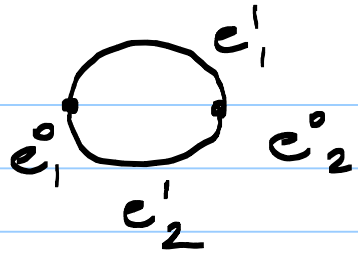
W : weak topology

If $X = X^n$ then we say that X has dimension n.

<u>Examples</u> 1) S^0 $\cdot \quad \cdot$ $(S^0 \subseteq \mathbb{R},\ x^2 = 1 \Rightarrow x = \pm 1)$

2) $S^1 \subseteq \mathbb{R}^2 \quad x^2 + y^2 = 1$





$\varphi_1 : \partial D_1^1 \longrightarrow X^0 = S^0 = \{e_1^0, e_2^0\}$

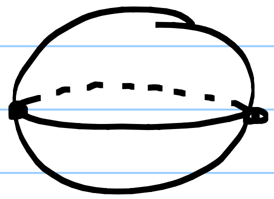
$\underset{D_1^1}{\overset{-1 \qquad 1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv}}$, $\partial D_1^1 = \{-1, 1\}$

$\varphi_1(-1) = e_1^0$, $\varphi_1(1) = e_2^0$.

$\varphi_2 : \partial D_2^1 \longrightarrow X^0 = S^0 = \{e_1^0, e_2^0\}$

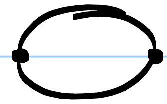
$\underset{D_2^1}{\overset{-1 \qquad 1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv}}$, $\partial D_2^1 = \{-1, 1\}$, $\varphi_2(-1) = e_1^0$, $\varphi_2(1) = e_2^0$.

3) $X = S^2$

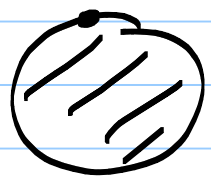


$X^0 = \{e_1^0, e_2^0\}$

$X^1 = \{e_1^0, e_2^0, e_1^1, e_2^1\}$



$X = X^2 = X^1 \dot\cup D_1^2 \dot\cup D_2^2 / \sim$



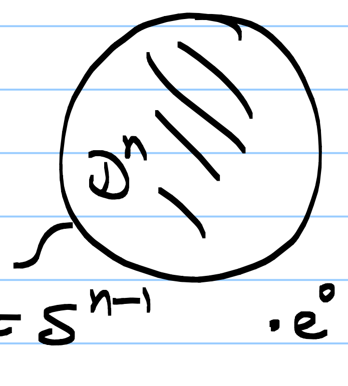
$\varphi_1 : \partial D_1^2 = S^1 \longrightarrow X^1 = S^1$

$\varphi_2 : \partial D_2^2 = S^1 \longrightarrow X^1 = S^1$

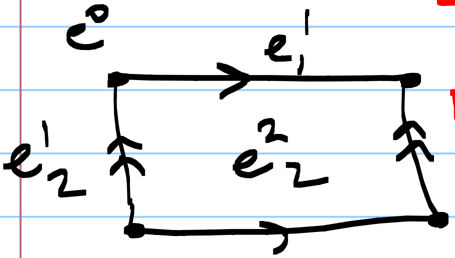
4) $S^\infty = \bigcup_{n=0}^{\infty} S^n$ (We'll see later that S^∞ is contractible, while no S^n is contractible.)

5) $S^n = X = X^0 \amalg D^n$

$X^0 = \{e^0\}$ $\quad \varphi : \underset{S^{n-1}}{\overset{\shortparallel}{\partial D^n}} \longrightarrow X^{n-1} = X^0$



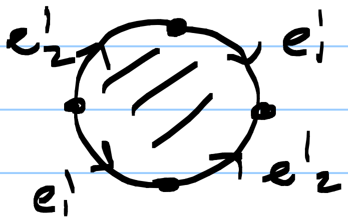
6) $T^2 = X = \underbrace{\underbrace{\underbrace{\{e^0\}}_{X^0} \cup e^1_1 \cup e^1_2}_{X^1} \cup e^2_2}_{X^2}$

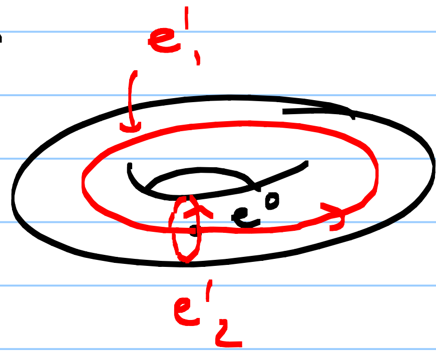


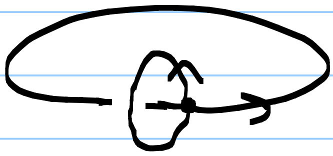
$\varphi^1_1 : \partial D^1 = \{\pm 1\} \longrightarrow X^0 = \{e^0\}$

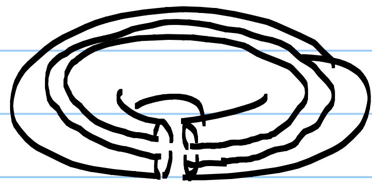
$\varphi^1_2 : \partial D^1 = \{\pm 1\} \longrightarrow X^0 = \{e^0\}$

$\varphi^2 : \partial D^2 = S^1 \longrightarrow X^1$

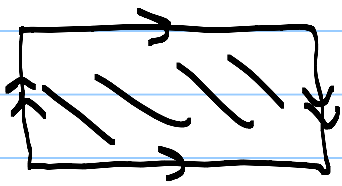




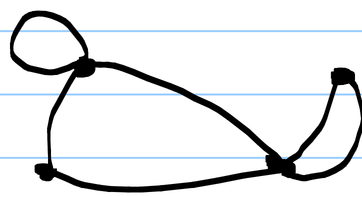




7) KB



8) One dimensional CW-complexes are called graphs.



X^0: the set of vertices

One dimensional cells are the edges.

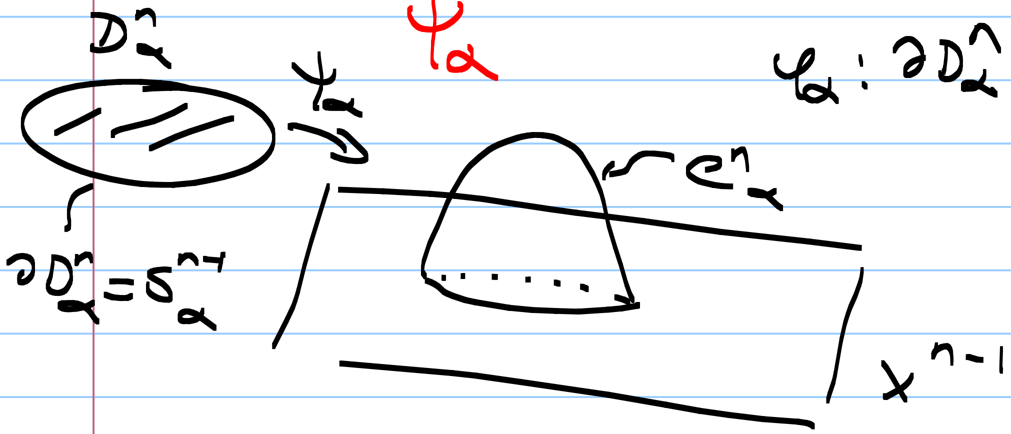
Video 10

<u>Characteristic map</u>: For any n-cell e^n_α of a CW-complex X the map

$\Psi_\alpha : D^n_\alpha \longrightarrow X^n$ is called characteristic map.

$$\begin{tikzcd} D^n_\alpha \arrow[r, hook] \arrow[rr, bend right, "\Psi_\alpha"'] & X^{n-1} \coprod_\alpha D^n_\alpha \arrow[r] & X^{n-1} \coprod_\alpha D^n_\alpha \big/ x \sim \varphi_\alpha(x) \end{tikzcd}$$

$\varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$

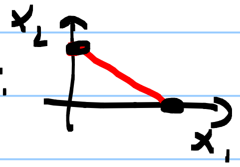


<u>Definition</u>: 1) If each characteristic map $\Psi_\alpha : D^n_\alpha \to X$ is an embedding then the CW-complex X is called regular.

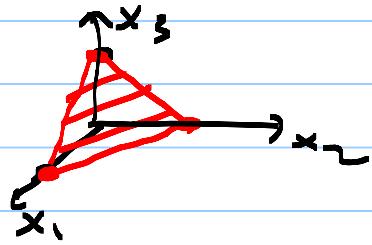
2) A simplicial complex is a CW-complex, where each D^n_α is Δ^n and the gluing maps are linear homeomorphisms:

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \ \sum_{i=1}^{n+1} x_i = 1\}.$$

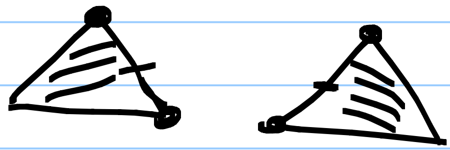
Δ^1:



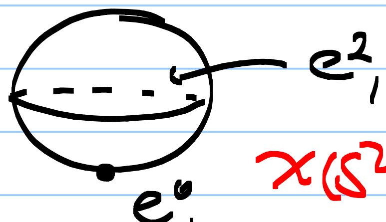
Δ^2:



Each Δ^n is homeomorphic to D^n.

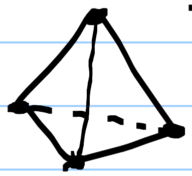


Example: $S^2 = e^2_1 \cup e^0_1$



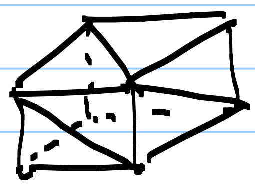
$\chi(S^2) = 1+1 = 2$

S^2: tetrahedron



This is a simplicial complex consisting of 4 0-simplices, 6 1-simplices and 4 2-simplices. $\chi(S^2) = 4-6+4 = 2$

S^2: cube



Simplicial complex having

8 0-simplices

18 1-simplices

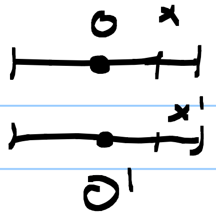
12 2-simplices

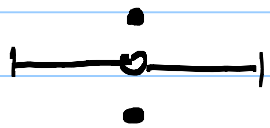
$\chi(S^2) = 8-18+12 = 2$

Definition: Euler characteristic of a finite cell complex X is defined to be the alternating sum of number of simplices:

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n \,, \quad c_n = \# \text{ of } n\text{-simplices of } X$$

Theorem: If two finite cell complexes are homeomorphic then their Euler characteristics are the same.

Example: X :  $/ x \sim x', x \neq 0$

X :  X has 4 0-cells,
2 1-cells.

$\chi(X) = 4 - 2 = 2.$

The Fundamental Group:

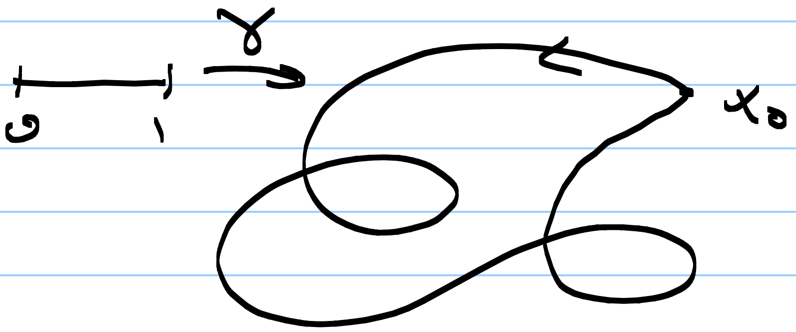
Fundamental group can be thought as a functor from the category of topological spaces with base points (continuous maps as morphisms) to the category of groups (homomorphisms as morphisms).

X topological space, $x_0 \in X$

$(X, x_0) \longrightarrow \pi_1(X, x_0)$

Definition: Given a based topological space (X, x_0) let $\mathcal{L}$ be the set of all loops at x_0:

$$\mathcal{L} = \{ \gamma : [0,1] \to X \mid \gamma \text{ continuous}, \gamma(0) = x_0 = \gamma(1) \}$$



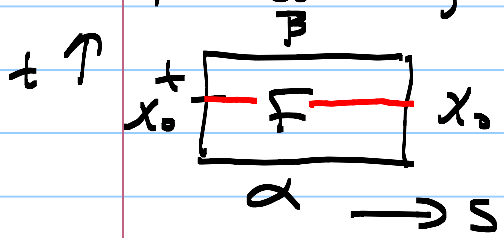
Define a homotopy relation on $\mathcal{L}$ as follows:

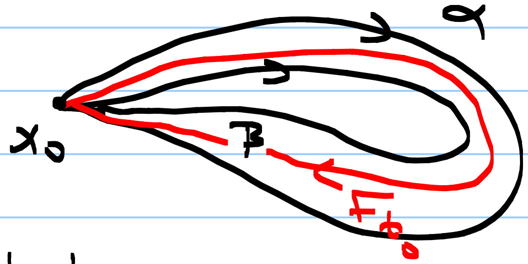
If $\alpha, \beta \in \mathcal{L}$, then $\alpha \sim \beta$ if and only if there is a homotopy

$$F : [0,1] \times [0,1] \longrightarrow X \quad \text{so that}$$

$F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$, $F(0,t) = x_0 = F(1,t)$,

for all $s, t \in [0,1]$.





$F_{t_0} = F(s, t_0)$

We've seen before that being homotopic is an equivalence relation.

Define $\pi_1(X, x_0)$ as the set of all equivalence classes of this relation: homotopy classes of based loops at x_0.

If α is a loop at x_0 then its homotopy class will be denoted as $[\alpha]$.

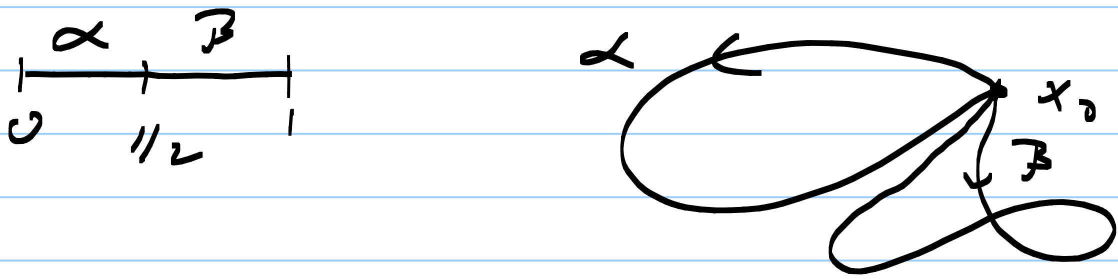
$$\pi_1(X, x_0) \doteq \mathcal{L}/\sim.$$

We define the group operation on $\pi_1(X, x_0)$ as follows:

If $[\alpha], [\beta] \in \pi_1(X, x_0)$ then let

$[\alpha]\cdot[\beta] \doteq [\alpha\cdot\beta]$, where

$$\alpha\cdot\beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1. \end{cases}$$

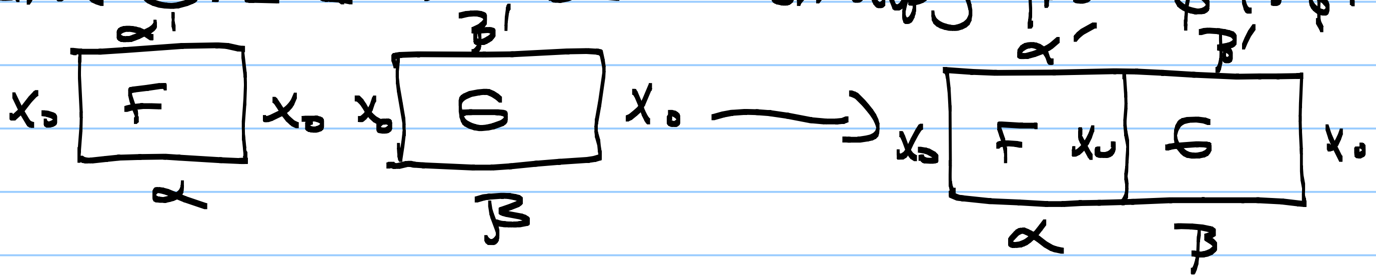


Need to show that this operation is well-defined. Let $\alpha, \alpha', \beta, \beta'$ be loops at x_0 with

$[\alpha] = [\alpha']$ and $[\beta] = [\beta']$.

<u>must show:</u> $[\alpha\cdot\beta] = [\alpha'\cdot\beta']$.

Let $F: I\times I \to X$ be a homotopy from α to α' and $G: I\times I \to X$ be a homotopy from β to β'.



$$H \doteq F\cdot G, \quad H(s,t) = \begin{cases} F(2s,t), & 0 \le s \le 1/2 \\ G(2s-1,t), & 1/2 \le s \le 1. \end{cases}$$

Then H is continuous by the Pasting Lemma and $H(s,0) = \alpha\cdot\beta$ and $H(s,1) = \alpha'\cdot\beta'$.

Hence, this operation on $\pi_1(X,x_0)$ is well defined.

The identity element of $\pi_1(X,x_0)$ is defined to be the homotopy class of the constant loop at x_0:

$$e : [0,1] \longrightarrow X, \quad e(s) = x_0, \ \forall s \in [0,1].$$

$$e = [e].$$

We must prove the following:

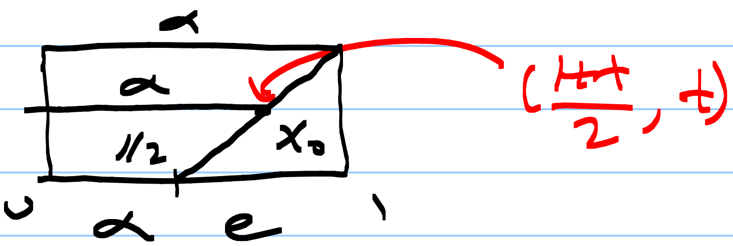
1) $e \cdot [\alpha] = [\alpha] = [\alpha] \cdot e$

2) For any $[\alpha]$ there is some $[\beta] \in \pi_1(X,x_0)$ so that $[\alpha]\cdot[\beta] = e = [\beta]\cdot[\alpha]$.

3) For any $[\alpha], [\beta], [\gamma] \in \pi_1(X,x_0)$ we must have

$$([\alpha]\cdot[\beta])\cdot[\gamma] = [\alpha]\cdot([\beta]\cdot[\gamma]).$$

Proof 1)



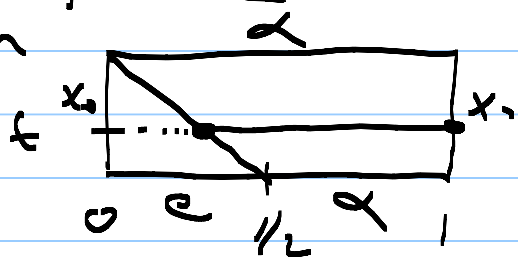
$\alpha(s \cdot \frac{2}{1+t})$ $\quad s=0, \ \alpha(0) = x_0,$
$\quad s = \frac{1+t}{2}, \ \alpha(1) = x_0$

Let $F(s,t) = \begin{cases} \alpha\left(\frac{2s}{1+t}\right), & 0 \le s \le \frac{1+t}{2}, \\ x_0, & \frac{1+t}{2} \le s \le 1. \end{cases}$

$$F(s,0) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ x_0, & 1/2 \le 1 \end{cases} = \alpha \cdot e$$

$$F(s,1)=\begin{cases}\alpha(s), & 0\le s\le 1\\ x_0, & 1\le s\le 1.\end{cases} = \alpha.$$

For the part $[e\cdot\alpha]=\alpha$ we use the diagram

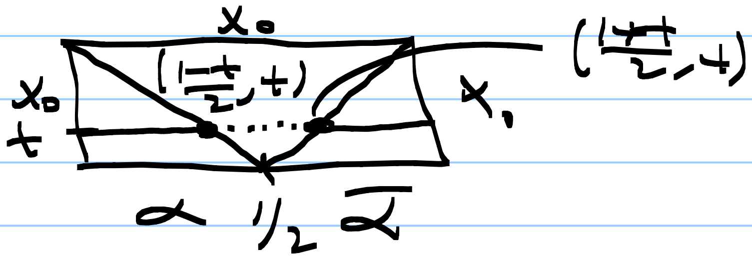


$F(s,t)=?$

2) Given $[\alpha]\in\pi_1(X,x_0)$ let $[\alpha]^{-1}$ as $[\bar{\alpha}]$, where

$$\bar{\alpha}(s)\doteq\alpha(1-s),\quad s\in[0,1].$$

<u>must show:</u> $[\alpha\cdot\bar{\alpha}]=e$.



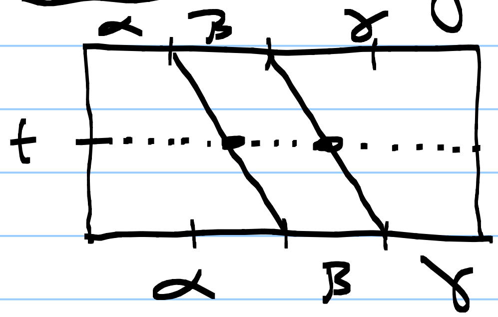
$$F(s,t)=\begin{cases}\alpha(2s), & 0\le s\le \frac{1-t}{2}\\ \alpha(1-t), & \frac{1-t}{2}\le s\le\frac{1+t}{2}\\ \alpha(2-2s), & \frac{1+t}{2}\le s\le 1.\end{cases}$$

$$F(s,0)=\begin{cases}\alpha(2s), & 0\le s\le 1/2\\ \alpha(1) & 1/2\le s\le 1/2\\ \alpha(2-2s) & 1/2\le s\le 1.\end{cases}$$

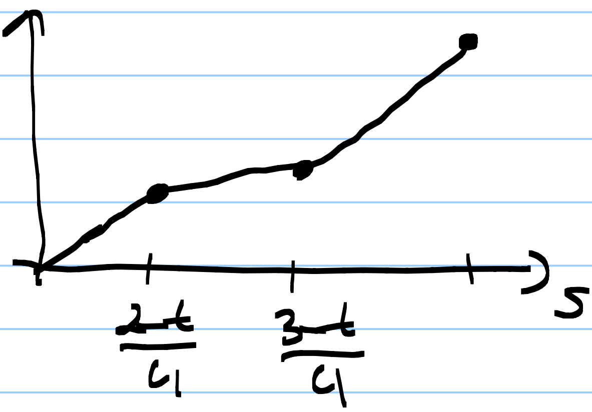
$$F(s,1)=\begin{cases}\alpha(2s), & 0\le s\le 0\\ \alpha(0), & 0\le s\le 1\\ \alpha(0), & 1\le s\le 1\end{cases} = e$$

Exercise: $[\bar{\alpha}\cdot\alpha]=[e]$.

3) Associativity $([\alpha]\cdot[\beta])\cdot[\gamma]=[\alpha]\cdot([\beta]\cdot[\gamma])$.



$$\ell_t(s)=\begin{cases}\dfrac{4s}{2-t}, & 0\le s\le \dfrac{2-t}{4}\\[2mm] 4s+t-1, & \dfrac{2-t}{4}\le s\le \dfrac{3-t}{4}\\[2mm] \dfrac{4s+3t-1}{1+t}, & \dfrac{3-t}{4}\le s\le 1.\end{cases}$$



Let
$$H(s,t)=\begin{cases}\alpha(\ell_t(s)), & 0\le s\le \dfrac{2-t}{4}\\[2mm] \beta(\ell_t(s)-1), & \dfrac{2-t}{4}\le s\le \dfrac{3-t}{4}\\[2mm] \gamma(\ell_t(s)-2), & \dfrac{3-t}{4}\le s\le 1.\end{cases}$$

Details are left as an exercise.

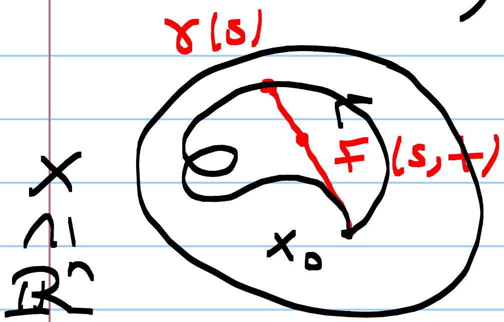
Video 11

Example For a convex subset X of $\mathbb{R}^n$ and any point $x_0 \in X$, $\pi_1(X, x_0) = (e)$, the trivial group.

Proof: Let $\gamma : [0,1] \longrightarrow X$ be a loop at x_0.

Then by the line homotopy γ is homotopic to the constant loop e at x_0:

$$F : X \times I \longrightarrow X, \quad F(s,t) = (1-t)\gamma(s) + t\,x_0$$



$F(s,0) = \gamma(s)$, $s \in [0,1]$
$F(s,1) = x_0$, $s \in [0,1]$.

$F(0,t) = (1-t)\gamma(0) + t x_0 = x_0$ and
$F(1,t) = (1-t)\gamma(1) + t x_0 = x_0$.

Hence, $[\gamma] = [e] = e$. ∎

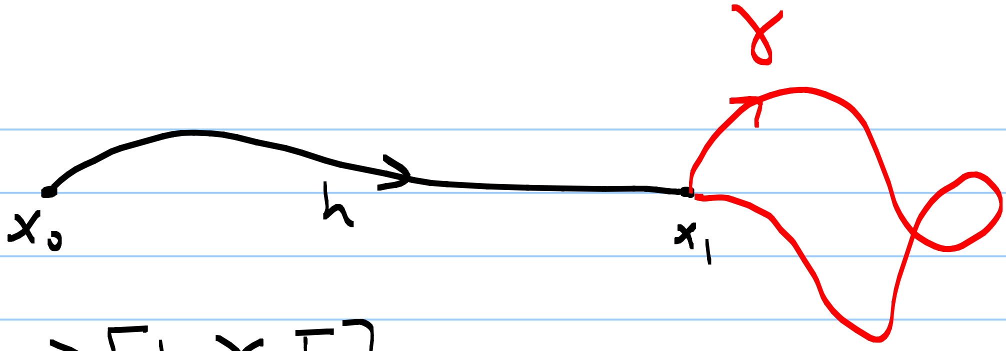
Exercise: If X is a contractible space to a point x_0, then $\pi_1(X, x_0) = (e)$.

Proposition: Let X be a topological space, $x_0, x_1 \in X$ points in X and $h : [0,1] \to X$ is a path joining x_0 to x_1: $h(0) = x_0$, $h(1) = x_1$. Then the map

$$\beta_h : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1) \text{ defined by}$$

$\beta_h([\gamma]) = [h \cdot \gamma \cdot \bar{h}]$ is a group isomorphism

Proof:



$$[\gamma] \longmapsto [h\cdot\gamma\cdot\bar{h}].$$

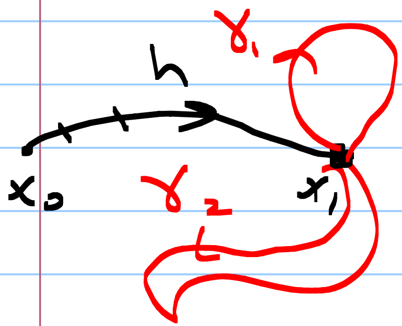
$h:[0,1]\to X$, $h(0)=x_0$, $h(1)=x_1$

$\bar{h}:[0,1]\to X$, $\bar{h}(s)=h(1-s)$, $\bar{h}(0)=h(1)=x_1$ and $\bar{h}(1)=h(0)=x_0$.

<u>β_h is a homomorphism:</u> Let $\gamma_i:[0,1]\to X$ be loops at x_1. Then $\gamma_1\cdot\gamma_2$ is also a loop at x_1, given by
$$(\gamma_1\cdot\gamma_2)(s)=\begin{cases}\gamma_1(2s), & 0\le s\le 1/2\\ \gamma_2(2s-1), & 1/2\le s\le 1.\end{cases}$$

$\beta_h([\gamma_1])=[h\cdot\gamma_1\cdot\bar{h}]$, $\beta_h([\gamma_2])=[h\cdot\gamma_2\cdot\bar{h}]$

$$\beta_h([\gamma_1])\cdot\beta_h([\gamma_2])=[h\cdot\gamma_1\cdot\bar{h}]\cdot[h\cdot\gamma_2\cdot\bar{h}]$$

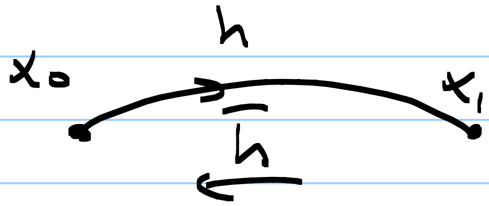


$$=[h\cdot\gamma_1\cdot\bar{h}\cdot h\cdot\gamma_2\cdot\bar{h}]$$
$$=[h\cdot\gamma_1\cdot e_{x_1}\cdot\gamma_2\cdot\bar{h}]$$
$$=[h\cdot(\gamma_1\cdot\gamma_2)\cdot\bar{h}]$$
$$=\beta_h([\gamma_1\cdot\gamma_2])$$

Hence, β_h is a group homomorphism.

<u>Claim:</u> $\beta_{\bar{h}}$ is the inverse of the homomorphism β_h.

Proof:



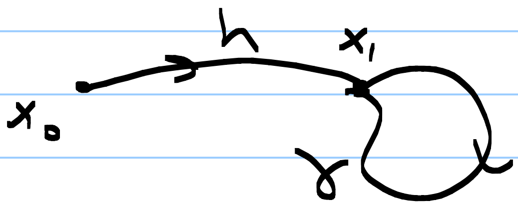
$\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $\beta_{\bar{h}} : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

Let $[\gamma] \in \pi_1(X, x_1)$. Then

$$(\beta_{\bar{h}} \circ \beta_h)([\gamma]) = \beta_{\bar{h}}([h \cdot \gamma \cdot \bar{h}])$$

$$= [\bar{h} \cdot (h \cdot \gamma \cdot \bar{h}) \cdot \bar{\bar{h}}] \quad (\bar{\bar{h}} = h)$$

$$= [(\bar{h} \cdot h) \cdot \gamma \cdot (\bar{h} \cdot h)]$$



$$= [(\bar{h} \cdot h)] \cdot [\gamma] \cdot [\bar{h} \cdot h]$$

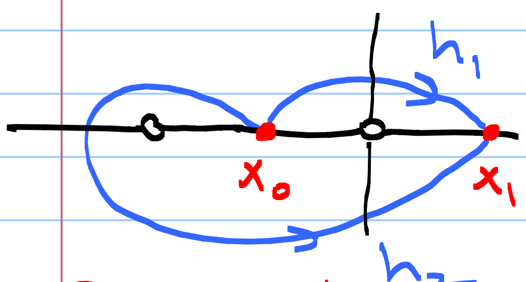
$$= e_{x_1} \cdot [\gamma] \cdot e_{x_1}$$

$$= [\gamma].$$

This finishes the proof. ∎

Remark: The isomorphism β_h from $\pi_1(X, x_1)$ to $\pi_1(X, x_1)$ is <u>not</u> canonical.

Example: $X = \mathbb{R}^2 \setminus \{(0,0), (-1,0)\}$



$\pi_1(X, x_0) \simeq F_2 \simeq \pi_1(X, x_1)$

β_{h_1} and β_{h_2} are <u>not</u> the same.

Exercise: If $\pi_1(X)$ is abelian then $\beta_{h_1} = \beta_{h_2}$ for any h_1, h_2

Video 12

Definition: A path connected space X is called simply connected if $\pi_1(X, x_0) = (e)$ for some (and thus all) $x_0 \in X$.

Proposition: A space X is simply connected if and only if for any two points x_0 and x_1 of X, there is a unique homotopy class of paths joining x_0 to x_1, where homotopies fix the end points at all times.

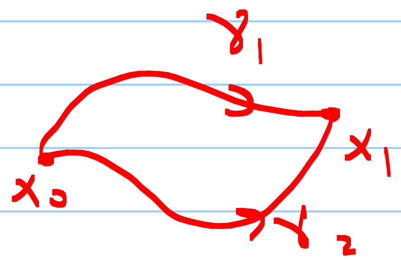
Proof: Exercise.

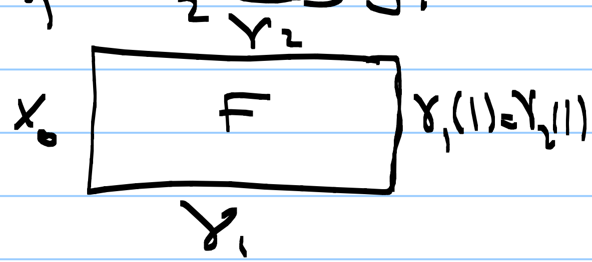
Remark: If X is simply connected space and $x_0 \in X$. Then there is a bijection between the set X and the set of homotopy classes of paths starting at x_0, where homotopies fix the end points.

$$\{\gamma : [0,1] \longrightarrow X \mid \gamma(0) = x_0\} / \sim$$

$$[\gamma] \longmapsto \gamma(1) \in X$$

$\gamma_1 \sim \gamma_2 \iff \exists F$





$$\varphi : \{\gamma : [0,1] \to X \mid \gamma(0) = x_0\} / \sim \;\longrightarrow X$$

$$[\gamma] \longmapsto \varphi([\gamma]) = \gamma(1).$$

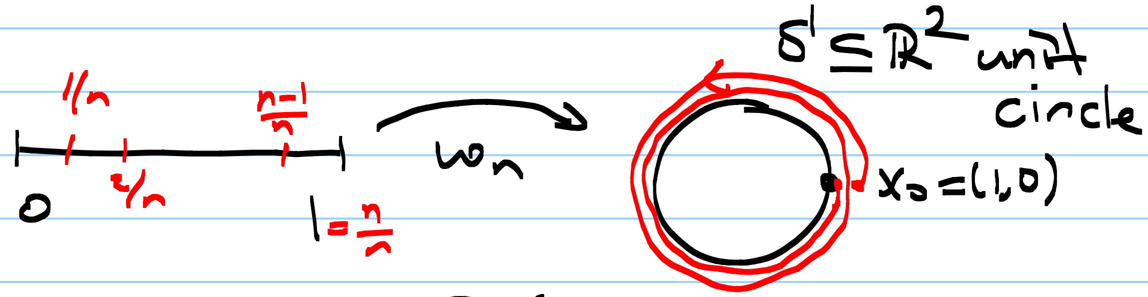
The Fundamental Group of the Circle

Theorem: The map $\Phi: \mathbb{Z} \to \pi_1(S^1, x_0)$, where $x_0 = (1,0)$ sending an integer n to the homotopy class of the loop

$$\omega_n : [0,1] \to S^1, \quad \omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$$

based at $x_0 = (1,0)$ is an isomorphism.

Proof:

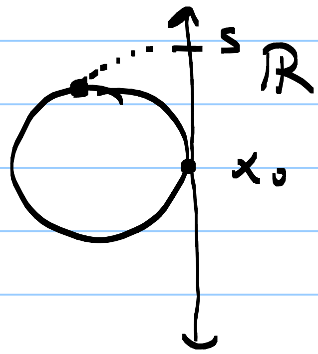


$$\mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$$
$$n \longmapsto [\omega_n].$$

The proof has several stages:

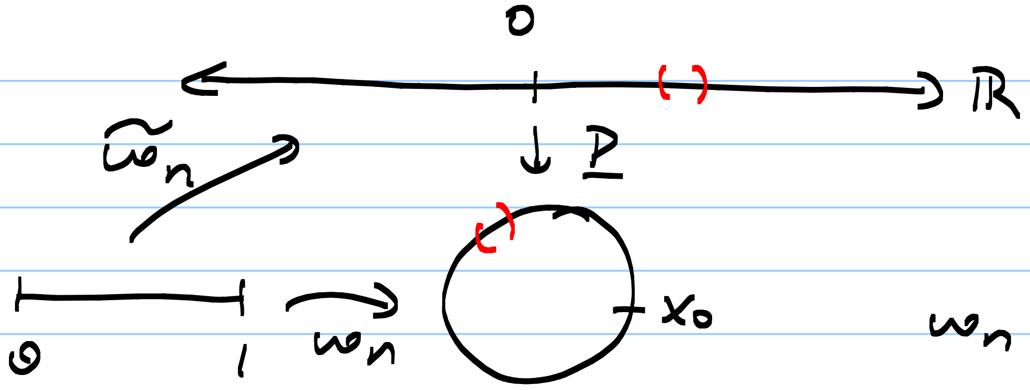
i) Consider the map $p: \mathbb{R} \to S^1$ given by

$p(s) = (\cos 2\pi s, \sin 2\pi s)$, $s \in \mathbb{R}$.



Let $\tilde{\omega}_n : [0,1] \to \mathbb{R}$ be given by $\tilde{\omega}_n(s) = ns$.

Note that $(p \circ \tilde{\omega}_n)(s) = p(\tilde{\omega}_n(s)) = p(ns) = \omega_n(s)$



$\omega_n = p \circ \tilde{\omega}_n$

Note that p is locally a homeomorphism.

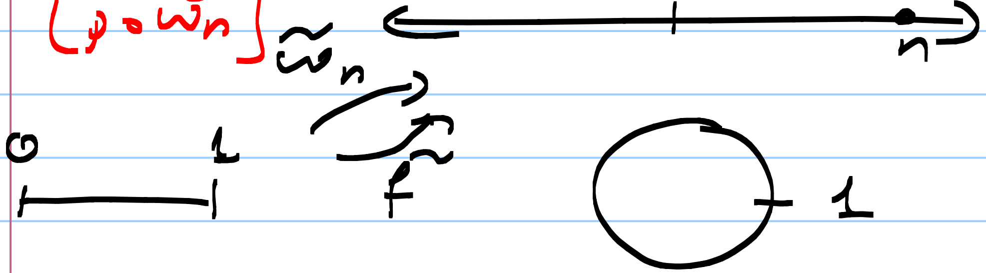
Note that $\Phi(n) = [\omega_n]$ can be defined as the homotopy class of the loop $p \circ \tilde{f}$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n, because any such $\tilde{f}$ is homotopic to $\tilde{\omega}_n$, keeping the end points fixed:

$t \longmapsto (1-t)\tilde{f} + t\,\tilde{\omega}_n$ and thus $p \circ \tilde{f}$ is homotopic to ω_n.

$$[p \circ \tilde{f}] = [\omega_n] = \Phi(n)$$

$\tilde{\omega}_n(0) = n = \tilde{f}(0)$

" $[p \circ \tilde{\omega}_n]$

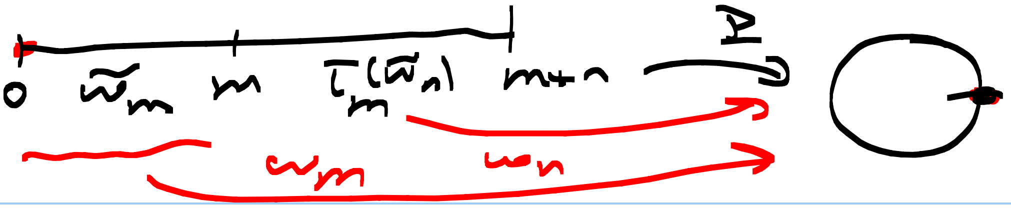


ii) <u>Claim</u>: Φ is a group homomorphism.

<u>Proof</u>: Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation map $\tau_m(x) = x + m$, $(m \in \mathbb{Z})$.

Then $\tilde{\omega}_m \cdot (\tau_m(\tilde{\omega}_n))$ is a path in $\mathbb{R}$ from 0 to $m+n$, so that $\Phi(m+n)$ is the homotopy

Video 13



class of $p(\tilde{\omega}_m \cdot (T_m(\tilde{\omega}_n))$.

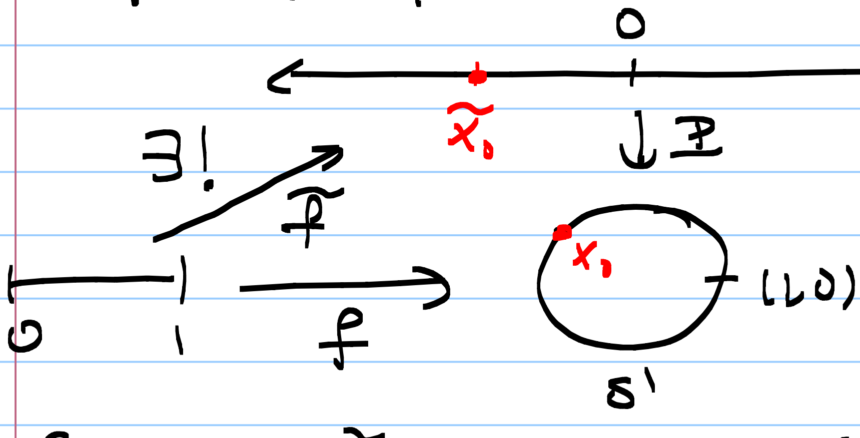
The image of the path $\tilde{\omega}_m \cdot (T_m(\tilde{\omega}_n))$ under p is just $\omega_m \cdot \omega_n$, so that

$$\Phi(m+n) = [\omega_m \cdot \omega_n] = [\omega_m] \cdot [\omega_n] = \Phi(m) \cdot \Phi(n).$$

Hence, Φ is a group homomorphism.

Next we'll show that Φ is a isomorphism.
To do so we'll use two facts:

a) For each path $f: I \to S^1$ starting a point x_0 and each $\tilde{x}_0 \in \mathbb{R}$ with $p(\tilde{x}_0) = x_0$ there is a unique lift $\tilde{f}: I \to \mathbb{R}$ starting at $\tilde{x}_0$.



$f(0) = x_0$, $\tilde{f}(0) = \tilde{x}_0$ and $p(\tilde{f}(s)) = f(s)$

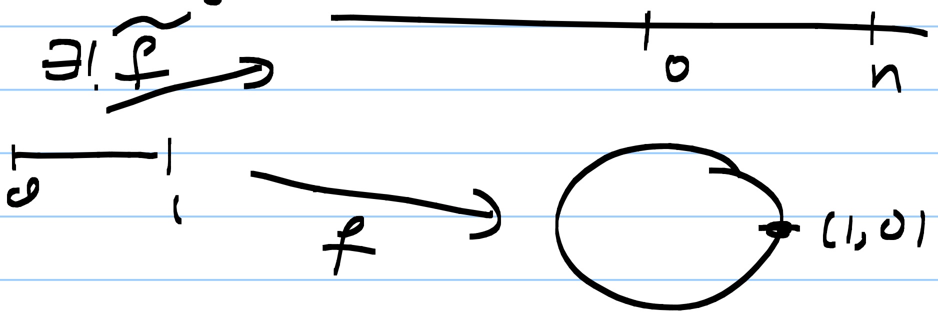
b) For each homotopy $f_t: I \to S^1$ of paths ($F: I \times I \to S^1$, $F(s,t) = f_t(s)$) stating at x_0 then there is a unique homotopy $\tilde{f}_t: I \to \mathbb{R}$ starting at $\tilde{x}_0$ and $p(\tilde{f}_t(s)) = f(s)$, for all s.

$(\tilde{F} : I \times I \longrightarrow \mathbb{R},\ \tilde{F}(s,t) = \tilde{f}_t(0))$

iii) (a) and (b) prove the theorem.

<u>Φ is surjective</u>: Let $f : I \longrightarrow S^1$ be a loop at the base point $(1,0)$ x_0, representing an element of $\pi_1(S^1,(1,0))$.

Now by Fact (a) there is a (unique) lift $\tilde{f}$ starting at $\tilde{x}_0 = 0 \in \mathbb{R}$ $(p(0) = (1,0))$



$f(0) = f(1) = (1,0)$. Since $p(\tilde{f}(1)) = f(1) = (1,0)$ we see that $\tilde{f}(1) \in \mathbb{Z}$ because

$p^{-1}(1,0) = \mathbb{Z} \subseteq \mathbb{R}$. Say $n = \tilde{f}(1)$.

By the extended definition of Φ we have

$\Phi(n) = [p \circ \tilde{f}] = [f]$ so that we are done.

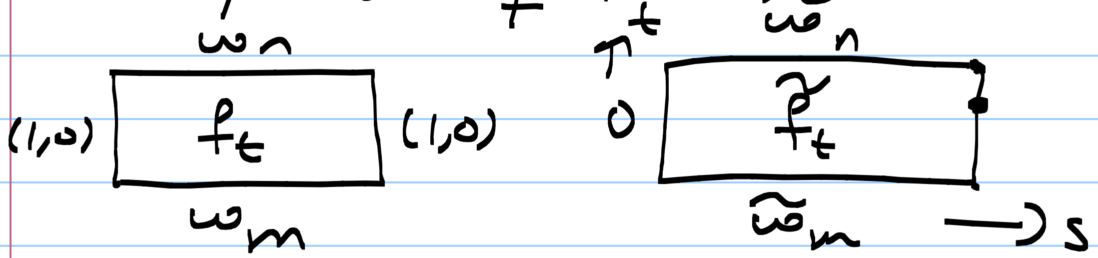
<u>Φ is injective</u>: Suppose that $\Phi(n) = \Phi(m)$, for some $m, n \in \mathbb{Z}$. So $[\omega_n] = [\omega_m]$ and thus ω_n and ω_m are homotopic.

<u>must show</u>: $m = n$.

Let f_t be a homotopy from $f_0 = \omega_m$ to $f_1 = \omega_n$. By (b) there is a unique lift $\tilde{f}_t$ of paths starting at 0.

The uniqueness part of (a) implies that

$\tilde{f}_0 = \widetilde{\omega}_m$ and $\tilde{f}_1 = \widetilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths the endpoints $\tilde{f}_t(1)$ is independent of t.



This is because $\tilde{F}(1,t) = \tilde{f}_t(1) \in \mathbb{Z}$ for all $t \in [0,1]$. and thus it must be a fixed integer.

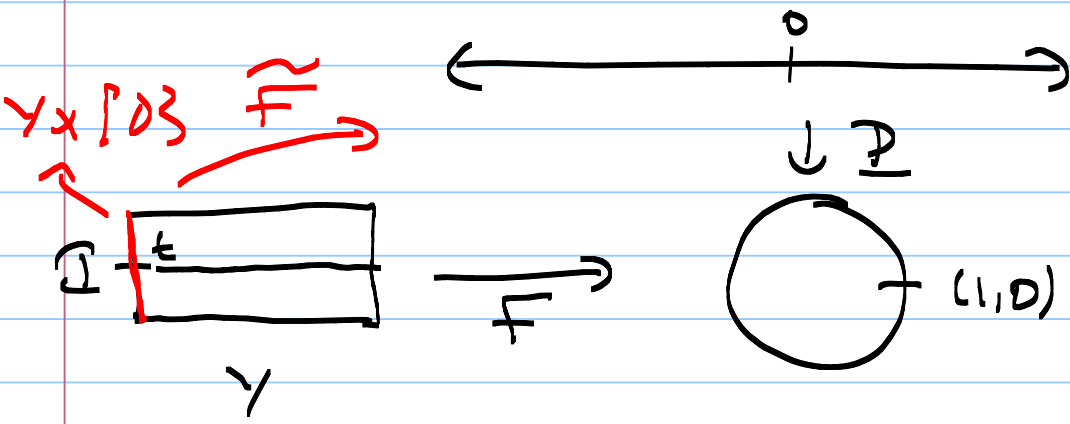
$\tilde{f}_t(1) = ?$ For $t=0$ the endpoints are m

and for $t=1$, the endpoints are n, and thus $m = n$.

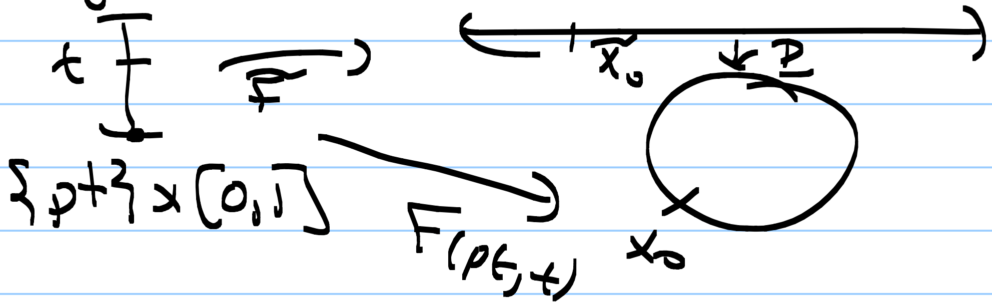
Now we must prove (a) and (b). Indeed, we'll prove another fact (c), which will imply both (a) and (b).

(c) Given a map $F : Y \times I \to S^1$ and another map $\tilde{F} : Y \times \{0\} \to \mathbb{R}$ lifting $F|_{Y \times \{0\}}$, then there is a unique map $\tilde{F} : Y \times I \to \mathbb{R}$ lifting F ($p \circ \tilde{F} = F$ as maps on $Y \times I$)

so that it restricts to the given $\tilde{F}$ on $Y \times \{0\}$.



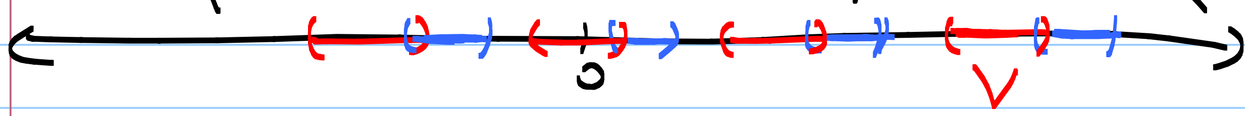
Taking $Y = \{pt\}$ (c) reduces to (a).

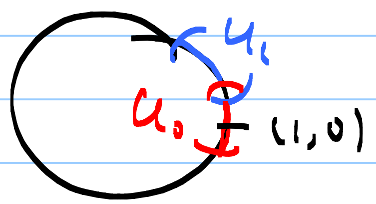


Similarly, taking $Y = [0,1]$ (c) becomes (b).

<u>To finish the proof we need to prove fact (c):</u>

To prove (c) we'll often use the following property of the map $P: \mathbb{R} \to S^1$: There is an open cover $\{U_\alpha\}$ of S^1 so that $P^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\mathbb{R}$ each of which is homeomorphic to U_α by P.





$P: V \to U_0$ homeomorphism

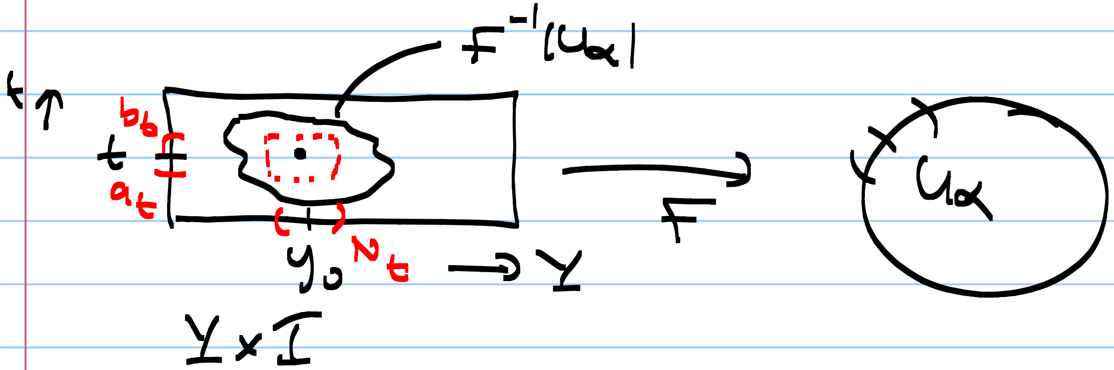
Definition: Let $P: X \to Y$ be an onto map. If Y has an open cover $\{U_\alpha\}$ so that for each α, the inverse image $P^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is homeomorphic to U_α via P, then we'll call the map $P: X \to Y$ a covering space/map.

Hence, the above $P: \mathbb{R} \to S^1$ is an example of a covering space (or map).

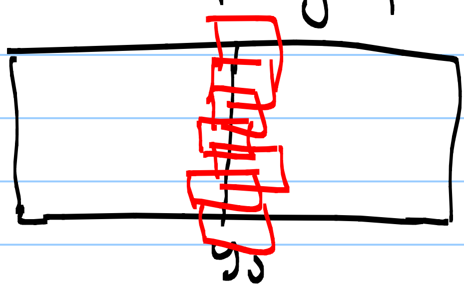
First let's construct a lift $\tilde{F}: N \times I \to \mathbb{R}$ for N some neighborhood in Y of a point $y_0 \in Y$.

Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ such that

$$F(N_t \times (a_t, b_t)) \subseteq U_\alpha \text{ for some } \alpha.$$



Since I is compact $\{y_0\} \times I$ is compact and thus $\{y_0\} \times I$ is covered by finitely many such products, say



$$N_{\alpha_i} \times (a_i, b_i)$$

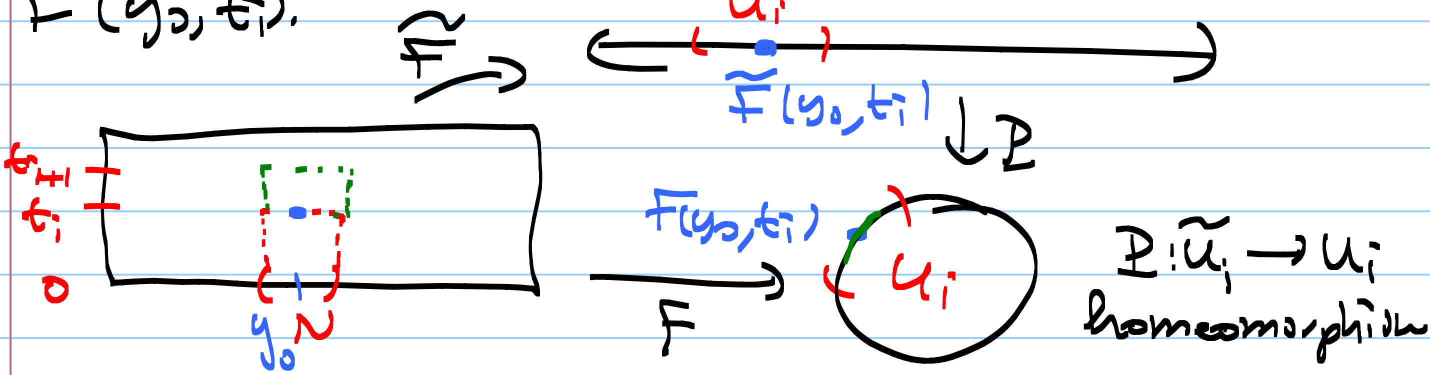
$i = 1, \ldots, k$.

Let $\mathcal{N} = \mathcal{N}_{\alpha_1} \cap \mathcal{N}_{\alpha_2} \cap \dots \cap \mathcal{N}_{\alpha_k}$. Also choose

$0 = t_0 < t_1 < t_2 < \dots < t_m = 1$ so that

$$F(\mathcal{N} \times [t_i, t_{i+1}]) \subseteq U_i \doteq U_{\alpha_i}, \quad i = 1, \dots, m.$$

Assume that $\tilde{F}$ has been constructed on $\mathcal{N} \times [0, t_i]$. Since $F(\mathcal{N} \times [t_i, t_{i+1}]) \subseteq U_i$ there some $\tilde{U}_i \subseteq \mathbb{R}$ projecting homeomorphically onto U_i by P and containing the point $\tilde{F}(y_0, t_i)$.



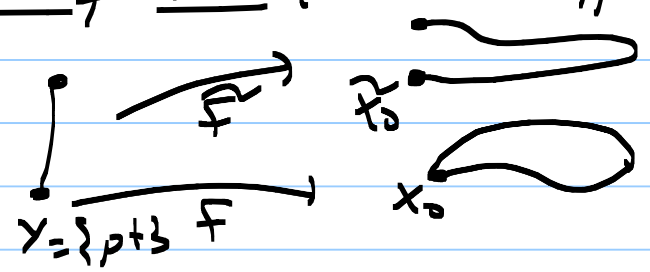
Now define $\tilde{F}$ on $\mathcal{N} \times [t_i, t_{i+1}]$ to be the composition of F with the inverse

$$P^{-1} : U_i \to \tilde{U}_i. \quad \text{So, } \tilde{F} = P^{-1} \circ F.$$

Repeating this finitely many times we get the required map $\tilde{F} : \mathcal{N} \times I \to \mathbb{R}$.

<u>Uniqueness of $\tilde{F}$ for $Y = \{pt\}$</u>

If



Since $Y = \{pt\}$ is a single point we may drop it from the notation.

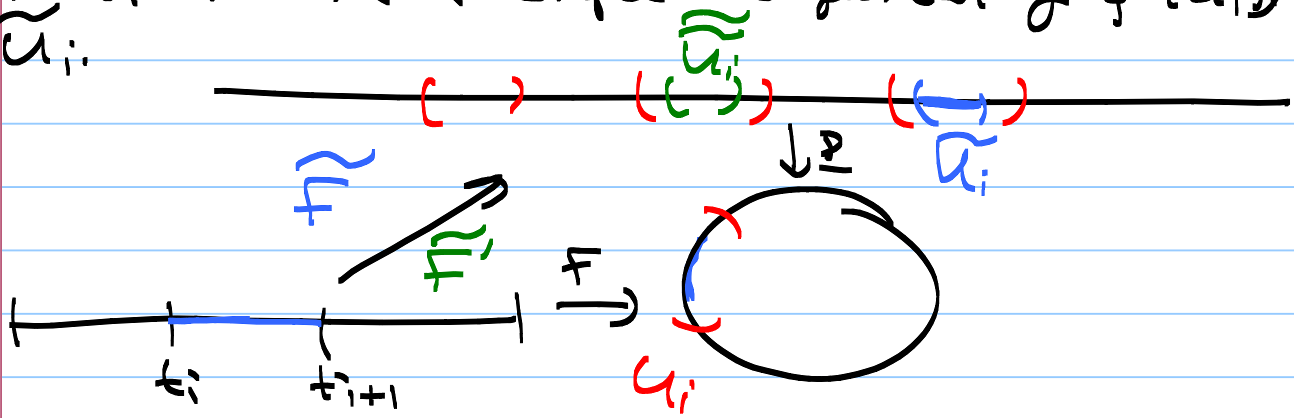
Suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of

$F: I \longrightarrow S^1$ so that $\tilde{F}(0) = \tilde{F}'(0)$. As above choose a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_m = 1 \text{ of } I$$

so that $F([t_i, t_{i+1}]) \subseteq U_i$ for some U_i.

Assume that $\tilde{F}$ and $\tilde{F}'$ agree on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected $\tilde{F}([t_i, t_{i+1}])$ must lie in a single component of $p^{-1}(U_i)$, say $\tilde{U}_i$.



Similarly, since $\tilde{F}'$ is also a lift $\tilde{F}'([t_i, t_{i+1}])$ must lie in a single component say $\tilde{\tilde{U}}_i$.

However, $\tilde{F}(t_i) = \tilde{F}'(t_i)$ and thus $\tilde{U}_i = \tilde{\tilde{U}}_i$

Since p is injective on $\tilde{U}_i$ and

$p \circ \tilde{F} = p \circ \tilde{F}'$ on we must have

$\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$. Inductively, we see that $\tilde{F} = \tilde{F}'$ on $[0, t_m] = [0, 1]$.

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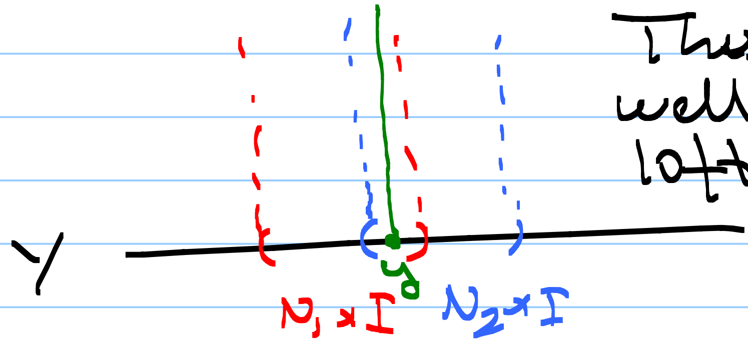
Finishing the Proof: Since $\tilde{F}$'s constructed above on the sets $N \times I$ are unique when restricted to segment $\{y\} \times I$, they must agree whenever two such $N \times I$'s overlap.

Thus we get a well-defined unique lift $\tilde{F}$ on $Y \times I$.



The unique lift $\tilde{F}: Y \times I \to \mathbb{R}$ is continuous since it is continuous on each $N \times I$.

This finishes the proof. ∎

So we have proved that the map

$$\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, (1,0)), \quad m \longmapsto [\omega_m], \; m \in \mathbb{Z},$$

is a group isomorphism.

Some Applications:

Theorem: (Fundamental Theorem of Algebra)

Every nonconstant polynomial with coefficient in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof: Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$

be a polynomial with $a_i \in \mathbb{C}$, $i=1,\dots,n$.
Assume on the contrary that $P(z)$ has no roots in $\mathbb{C}$. Then for any real number $r \geq 0$, the formula

$$f_r(s) = \frac{P(re^{2\pi i s})/P(r)}{|P(re^{2\pi i s})/P(r)|}, \quad f_r : [0,1] \longrightarrow S^1$$

a continuous function for each r. Indeed,

$F : [0,1] \times \mathbb{R}^{\geq 0} \longrightarrow S^1$, $F(s,r) = f_r(s)$ is a continuous function. Indeed, each f_r is loop on S^1 based at $(1,0) = 1 \in \mathbb{C}$,

$f_r(0) = 1$ and $f_r(1) = 1$, for all $r \geq 0$.

Hence, F defines a homotopy from f_0 to and f_r, $r \geq 0$.

$f_0 : [0,1] \longrightarrow S^1$, $f_0(s) = 1$, $\forall s \in [0,1]$, is the constant loop.

Hence, $[f_r] = [f_0] = e \in \pi_1(S^1, 1)$

Indeed, $e = 0$ in $\mathbb{Z} \cong \pi_1(S^1, 1)$.

Let $r = |a_1| + |a_2| + \dots + |a_n| + 1$. Now, for any $z \in \mathbb{C}$, with $|z| = r$, then

$|z^n| = r^n = r \cdot r^{n-1} > (|a_1| + \dots + |a_n|)\,|z|^{n-1}$

In particular,

$$|a_1 z^{n-1} + \cdots + a_{n-1} z + a_n| \leq |a_1 z^{n-1}| + \cdots + |a_{n-1} z| + |a_n|$$

$(r \geq 1)$

$$= |a_1| r^{n-1} + \cdots + |a_{n-1}| r + |a_n|$$

$$\leq |a_1| r^{n-1} + \cdots + |a_{n-1}| r^{n-1} + |a_n| r^{n-1}$$

$$= (|a_1| + \cdots + |a_n|) r^{n-1}$$

$$< |z^n|.$$

Hence, the polynomial

$$P_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1} z + a_n)$$

has no roots on the circle $|z| = r$, for any $0 \leq t \leq 1$.

Now the formula

$$\frac{P_t(re^{2\pi i s})/P_t(r)}{|P_t(re^{2\pi i s})/P_t(r)|}, \quad t \in [0,1], \text{ defines a}$$

homotopy.

$$t = 0 \Rightarrow \frac{(re^{2\pi i s})^n / r^n}{|e^{2\pi i s}|} = e^{2\pi i n s}, \quad s \in [0,1]$$

which is ω_n.

Hence, $n = [\omega_n] = [f_r] = 0$ in $\mathbb{Z} \simeq \pi_1(S^1, 1)$.

Video 16

Thus $P(z)$ is a polynomial of degree $n=0$, i.e., it is a constant polynomial, a contradiction to the assumption.
Therefore, $P(z)$ must have a zero in $\mathbb{C}$. ■

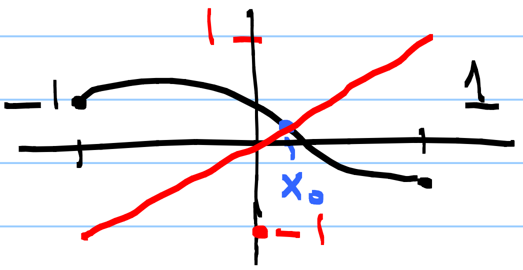
Theorem: Every continuous map $f: D^2 \to D^2$ has a fixed point.

Remark: Indeed the same holds for any $f: D^n \to D^n$.

$n=1$ is known from Intermediate Value Theorem.

$f: [-1,1] \longrightarrow [-1,1]$

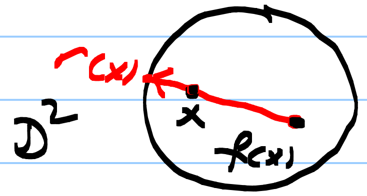
$f(x_0) = x_0$



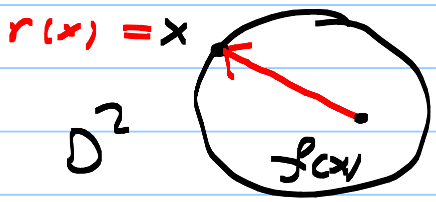
For $n \geq 3$ we'll Homology or Higher Homotopy theory.

Proof for $n=2$: (Brouwer proved this for D^n in 1910)

Assume on the contrary that the given function
$$f: D^2 \longrightarrow D^2$$
has no fixed points. Hence, $f(x) \neq x$, for all $x \in D^2$. Now define a retraction
$r: D^2 \longrightarrow S^1$ as follows: $r(x)$ is the intersection of $S^1 = \partial D^2$ with the ray starting at $f(x)$ and passing through x.



The continuity of $r(x)$ is left as an exercise.
Moreover, if $x \in S^1 = \partial D^2$, then $r(x) = x$.



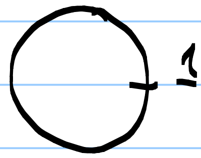
Hence, $r : D^2 \longrightarrow \partial D^2 = S^1$ is a retraction.

Now we need a fact:

<u>Claim</u>: There is no retraction from D^2 to the boundary $\partial D^2 = S^1$.

<u>Proof</u>: Suppose that there is a retraction $r : D^2 \longrightarrow S^1$.

$$\begin{array}{ccccc} S^1 & \overset{i}{\hookrightarrow} & D^2 & \overset{r}{\longrightarrow} & S^1 \\ x & \longmapsto & x & \longmapsto & r(x) \end{array} \quad , \; x \in S^1.$$



This gives homomorphism on π_1:

$$(r \circ i)(1) = r(i(1)) = r(1) = 1.$$

$$(S^1, \{1\}) \xrightarrow{\;i\;} (D^2, \{1\}) \xrightarrow{\;r\;} (S^1, \{1\})$$

<u>Fact</u>: If $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map of based topological spaces then the map

$$f_{\#} : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \; [\gamma] \longmapsto [f \circ \gamma],$$

is a group homomorphism.

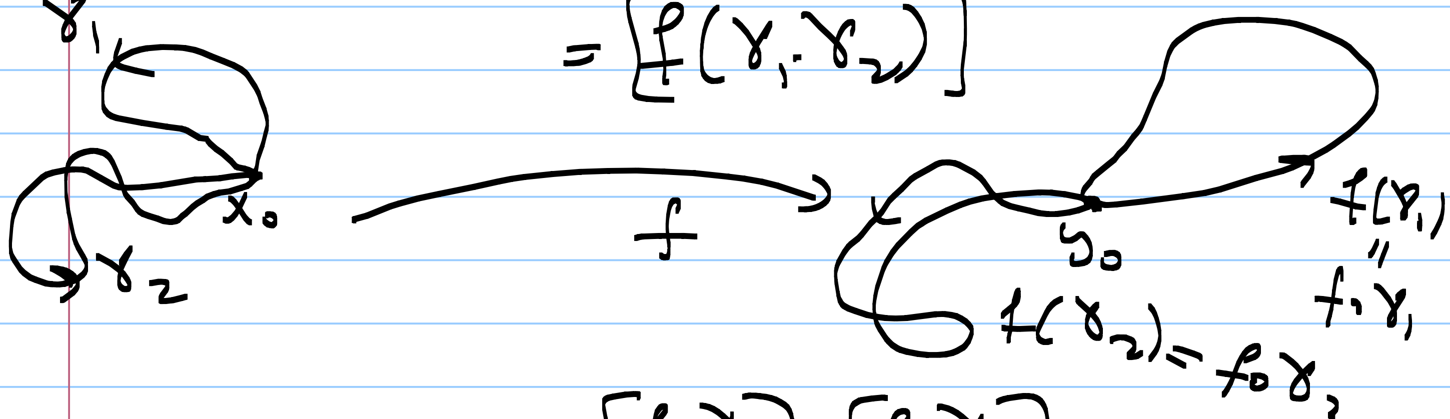
Proof of the fact:

i) It is well defined. If γ_t is a homotopy from γ_0 to γ_1 of based loops at x_0 then $f\circ\gamma_t$ is a homotopy of based loops at $y_0 = f(x_0)$ from $f\circ\gamma_0$ to $f\circ\gamma_1$.

ii) If $[\gamma_1]$ and $[\gamma_2]$ are two classes in $\pi_1(X,x_0)$ then

$$f_\#([\gamma_1]\cdot[\gamma_2]) = f_\#([\gamma_1\cdot\gamma_2])$$

$$= [f(\gamma_1\cdot\gamma_2)]$$



$$= [f\circ\gamma_1]\cdot[f\circ\gamma_2]$$

$$= f_\#([\gamma_1])\cdot f_\#([\gamma_2]).$$

Hence, $f_\#$ is a group homomorphism. ∎

Remark: If $f:(X,x_0)\to(X,x_0)$ is the identity function then the homomorphism

$$f_\#:\pi_1(X,x_0)\longrightarrow\pi_1(X,x_0) \text{ is the}$$

identity homomorphism.

Back to the proof of the Application:

$$(S^1,1) \overset{i}{\hookrightarrow} (D^2,1) \overset{r}{\longrightarrow} (S^1,1)$$

$$r \circ i = \mathrm{id}_{(S^1,1)}.$$

$$\begin{tikzcd}
\pi_1(S^1,1) \arrow[r, "i_\#"] \arrow[rr, bend right=40] \arrow[d, phantom, "\cong" rotate=90] & \pi_1(D^2,1) \arrow[r, "r_\#"] \arrow[d, phantom, "\parallel"] & \pi_1(S^1,1) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} \arrow[r] & (e) \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$$(r \circ i)_\# = \left(\mathrm{id}_{(S^1,1)}\right)_\# = \mathrm{id}_{\pi_1(S^1,1)}$$

This is a contradiction since the identity isomorphism of $\mathbb{Z}$ passes through the trivial group.

This finishes the proof. ∎

Theorem (Borsuk–Ulam Theorem)

For every continuous map $f: S^2 \to \mathbb{R}^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ there exists a pair of antipodal points x and $-x$ on S^2 so that $f(x) = f(-x)$.

air pressure, temp.

$$f(x) = (P(x), T(x))$$

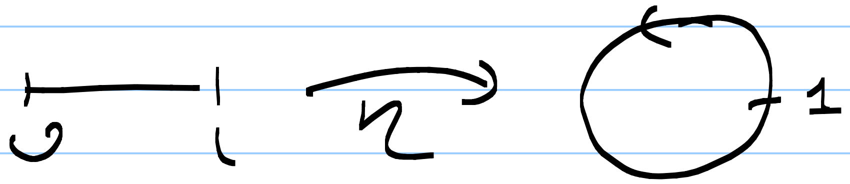
$x = (a,b,c)$, $-x = (-a,-b,-c)$

Proof: Suppose that $f(x) \neq f(-x)$, for all $x \in S^2$. We must arrive at a contradiction!

Define a map $g : S^2 \longrightarrow S^1$ as follows:

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}, \quad x \in S^2.$$

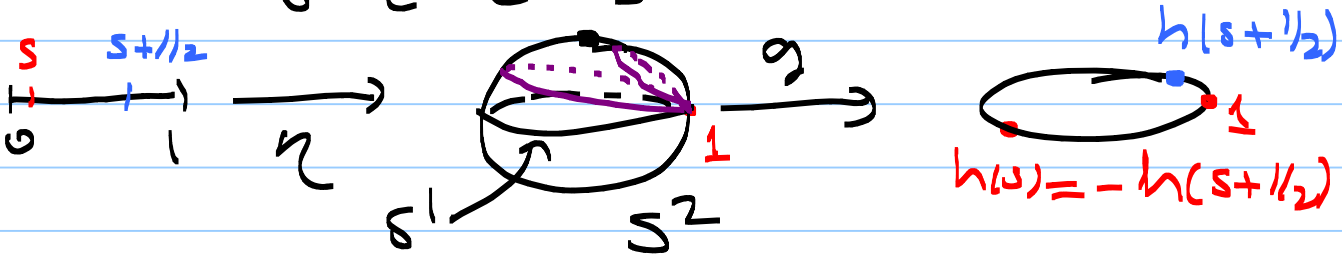
Let $\eta : [0,1] \longrightarrow S^1 \subseteq S^2$ by $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$



$[\eta] = [\omega_1] = 1$

$\pi_1(S^1, 1) \simeq \mathbb{Z}$

Let $h = g \circ \eta : [0,1] \longrightarrow S^1$



Note that $g(-x) = -g(x)$, for all $x \in S^2$.

Hence, $h(s+1/2) = g(\eta(s+1/2))$

$$= g(\cos 2\pi(s+1/2), \sin 2\pi(s+1/2))$$
$$= g(\cos(2\pi s + \pi), \sin(2\pi s + \pi))$$
$$= g(-\cos 2\pi s, -\sin 2\pi s)$$
$$= -g(\cos 2\pi s, \sin 2\pi s)$$
$$= -g(\eta(s))$$

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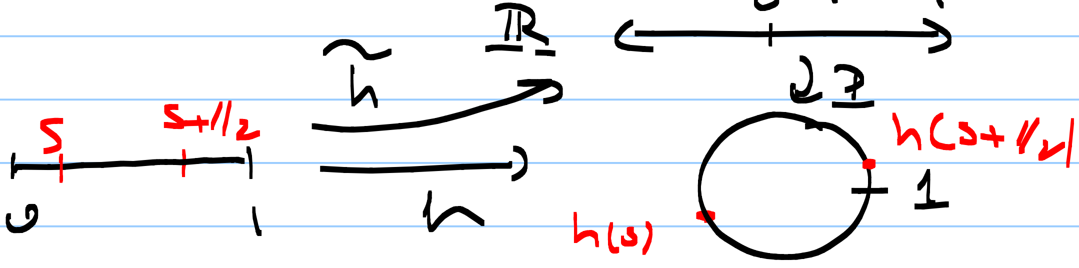
$= -h(s), \quad s \in [0,1].$

Let $\tilde{h} : [0,1] \to \mathbb{R}$ be the lift of h with $\tilde{h}(0) = 0$.



Since $h(s+1/2) = -h(s)$ we get

$$\tilde{h}(s+1/2) = \tilde{h}(s) + \frac{q}{2}$$ for some odd integer $q \in \mathbb{Z}$,

because $p(\tilde{h}(s+1/2)) = h(s+1/2) = -h(s) = -p(\tilde{h}(s))$

Note that since $q = 2(\tilde{h}(s+1/2) - \tilde{h}(s))$ and $\tilde{h}$ is continuous q must be independent of s.
In particular,

$$\tilde{h}(1) = \tilde{h}(1/2) + q/2 = \tilde{h}(0) + q \Rightarrow [h] = q \in \mathbb{Z} \cong \pi_1(S^1, 1).$$

Since q is an odd integer $[h] \neq 0 \in \mathbb{Z}$ and thus h is <u>not</u> null homotopic.
However, $h = g \circ \eta$ and η is clearly null homotopic. Thus h is null homotopic, and thus we arrived at a contradiction.
This finishes the proof. ∎

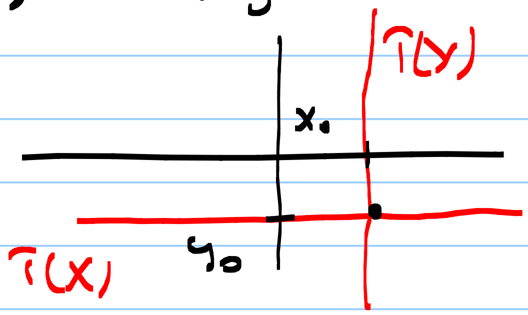
<u>Proposition</u>: Let (X, x_0) and (Y, y_0) be path connected based spaces. Then $(X \times Y, (x_0, y_0))$ is a path connected based space and

$\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof: $i : (X, x_0) \longrightarrow (X \times Y, (x_0, y_0))$, $x \longmapsto (x, y_0)$, $x \in X$

$j : (Y, y_0) \longrightarrow (X \times Y, (x_0, y_0))$, $y \longmapsto (x_0, y)$, $y \in Y$.

Clearly i and j are continuous maps.



We have also projection maps:

$P_X : (X \times Y, (x_0, y_0)) \longrightarrow (X, x_0)$, $(x, y) \longmapsto x$, $(x, y) \in X \times Y$,

and $P_Y : (X \times Y, (x_0, y_0)) \longrightarrow (Y, y_0)$, $(x, y) \longmapsto y$, $(x, y) \in X \times Y$.

$P_X \circ i : (X, x_0) \longrightarrow (X, x_0)$, $x \longmapsto (x, y_0) \longmapsto x$, $x \in X$,
so that $P_X \circ i = id_{(X, x_0)}$

and similarly $P_Y \circ j = id_{(Y, y_0)}$.

$i_\# : \pi_1(X) \longrightarrow \pi_1(X \times Y)$, $j_\# : \pi_1(Y) \longrightarrow \pi_1(X \times Y)$.

$P_{X\#}$ $\qquad\qquad$ $P_{Y\#}$

Consider the map
$\varphi : \pi_1(X) \times \pi_1(Y) \longrightarrow \pi_1(X \times Y)$ given by

$\varphi([f],[g]) = [h]$, where $h: I \to X \times Y$,
$h(s) = (f(s), g(s))$.

<u>must show:</u> i) φ is well defined
ii) φ is a homomorphism
iii) φ has an inverse

$$\varphi^{-1}: \pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$$

$$\varphi^{-1}([h]) = ([f], [g])$$

Rest is exercise!

<u>Corollary</u> $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ and in general

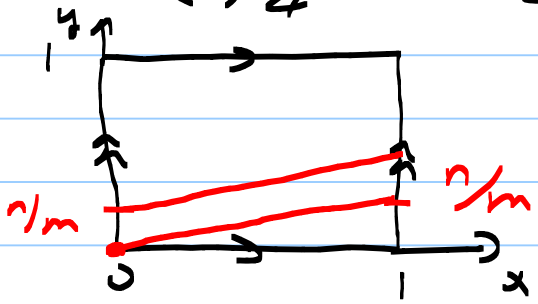
$$\pi_1(\underbrace{S^1 \times S^1 \times \dots \times S^1}_{n\text{-copies}}) \simeq \underbrace{\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}}_{n\text{-copies}}.$$

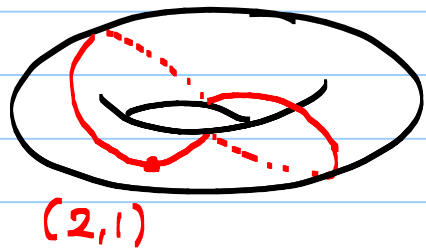
<u>Example:</u> $(m,n) \in \mathbb{Z} \times \mathbb{Z}$

$S^1 = \mathbb{R}/\mathbb{Z}$ $x \sim x+n$, $\forall n \in \mathbb{Z}$

$S^1 \times S^1 = (\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z})$

$(m,n) \longmapsto (e^{2\pi i s m}, e^{2\pi i s n})$





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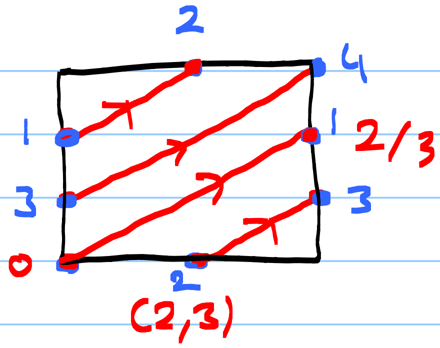
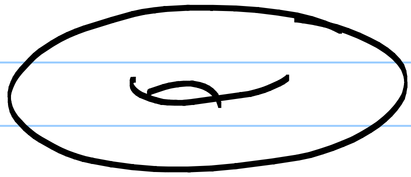
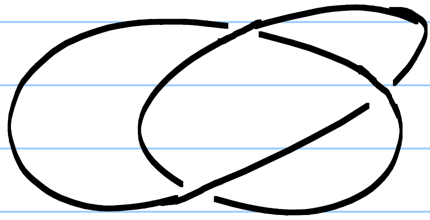






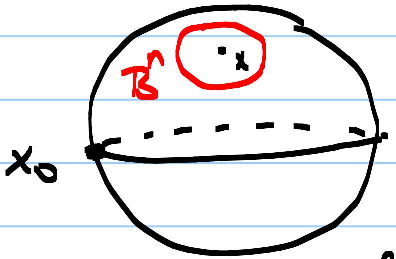
Trefoil Knot.

Exercise: The homotopy class of the loop corresponding to $(m,n) \in \mathbb{Z} \times \mathbb{Z} \simeq \pi_1(T^2)$ is represented by an embedded circle if and only if m and n are relatively prime.

Theorem: $\pi_1(S^n, x_0) = (e)$ if $n \geq 2$.

Proof: Let $f : [0,1] \to S^n$ be any loop based at x_0.

Let $x \in S^n$, $x \neq x_0$ and choose a small open ball B^n around. The inverse image of B^n, $f^{-1}(B^n)$ is a

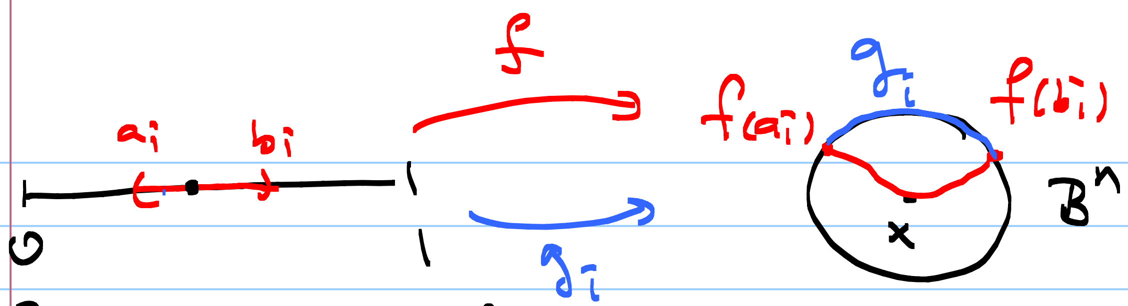


disjoint union of open intervals (a_i, b_i) in I. Since $f^{-1}(x)$ is a compact subset of $[0,1]$ and it is covered by the open intervals (a_i, b_i) $f^{-1}(x)$ is covered by finitely many (a_i, b_i). Now for each these finitely many intervals choose some

$$g_i : [a_i, b_i] \longrightarrow S^n \text{ so that } g_i \text{ is homotopic}$$

to $f_i|_{[a_i, b_i]}$, $f(a_i) = g(a_i)$, $f(b_i) = g(b_i)$ and

$g_i([a_i, b_i]) \subseteq \partial B^n$.



Replacing each $f|_{[a_i, b_i]}$, for finitely many i, by suitable g_i, we obtain the loop represented by g so that $[f] = [g]$ and $g(I) \subseteq S^n \setminus \{x\}$.

$g : [0,1] \longrightarrow S^n \setminus \{x\} \simeq \mathbb{R}^n$ is null homotopic and thus $[f] = [g] = e$ in $\pi_1(S^n, x_0)$.

This finished the proof. ∎

<u>Examples</u>: 1) $x \in \mathbb{R}^n$, then $\mathbb{R}^n \setminus \{x\} \simeq \mathbb{R}^{>0} \times S^{n-1}$ homeomorphic.

$$\varphi : \mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R}^{>0} \times S^{n-1}, \quad y \longmapsto \left(\|y - x\|, \frac{y-x}{\|x-y\|} \right),$$

$$\mathbb{R}^n \setminus \{x\} \simeq \mathbb{R}^{>0} \times S^{n-1} \simeq \mathbb{R} \times S^{n-1} \text{ homeomorphism.}$$
$$(t, p) \longmapsto (\ln t, p)$$

$$\pi_1(\mathbb{R}^n \setminus \{x\}) \simeq \pi_1(\mathbb{R} \times S^{n-1}) = \pi_1(\mathbb{R}) \times \pi_1(S^{n-1})$$
$$\simeq (e) \times \pi_1(S^{n-1})$$
$$\simeq \pi_1(S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ (e) & \text{if } n > 2 \end{cases}$$

<u>Corollary</u>: Hence $\mathbb{R}^2$ is <u>not</u> homeomorphic to any $\mathbb{R}^n$ if $n \neq 2$.

Proof: $n=1$ $\mathbb{R}^2$? $\mathbb{R}$

If $f:\mathbb{R}^2\to\mathbb{R}$ is a homeomorphism then $f:\mathbb{R}^2\setminus\{p\}\to\mathbb{R}\setminus\{f(p)\}$ would be still a homeomorphism. This is a contradiction because $\mathbb{R}^2\setminus\{p\}$ is connected and $\mathbb{R}\setminus\{f(p)\}$ is not.

$n>2$: $f:\mathbb{R}^2\to\mathbb{R}^n$. This time we would get a homeomorphism from $\mathbb{R}^2\setminus\{p\}$ to $\mathbb{R}^n\setminus\{f(p)\}$, where $\mathbb{R}^n\setminus\{f(p)\}$ is simply connected and $\mathbb{R}^2\setminus\{p\}$ is not simply connected.

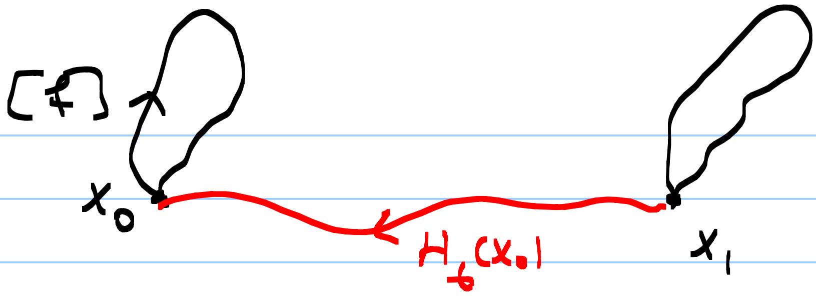
Proposition: Let $\varphi:X\to Y$ be a homotopy equivalence. Then $\varphi_\#:\pi_1(X,x_0)\to\pi_1(Y,\varphi(x_0))$ is an isomorphism, for any $x_0\in X$.

Proof: By the assumption there is a continuous map $\psi:Y\to X$ so that

$\psi\circ\varphi:X\to X$ is homotopic to id_X and

$\varphi\circ\psi:Y\to Y$ is homotopic to id_Y.

Let $x_1=(\psi\circ\varphi)(x_0)=\psi(\varphi(x_0))$



$$(\psi\circ\varphi)_\#([f])$$

$$\psi\circ\varphi \simeq Id_X$$

$$H: X\times I \longrightarrow X$$

$$H(x,t) = H_t(x)$$

$$H_0(x) = (\psi\circ\varphi)(x)$$

$$H_1(x) = Id_X(x) = x$$

The map $t \longmapsto H_t(x_0)$ is a path from x_1 to x_1

$$\pi_1(X,x_0) \xrightarrow[(\psi\circ\varphi)_\#]{} \pi_1(X,x_1) \xrightarrow[\beta_{H_t(x_0)}]{\cong} \pi_1(X,x_0)$$

$$\varphi_\#: \pi_1(X,x_0) \longrightarrow \pi_1(Y,\varphi(x_0))$$

$$\psi: \pi_1(Y,\varphi(x_0)) \longrightarrow \pi_1(X,x_1)$$

Rest is exercise.

Video 19

Van Kampen's Theorem:

Free Products of Groups: Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be family of groups. Then the free product of this family is defined to be group by means of the following universal property:

Notation: $\ast_\alpha G_\alpha$: Free product of G_α's.

$\mathcal{P}$: If $\varphi_\alpha : G_\alpha \to H$ is a homomorphism for each $\alpha\in\Lambda$, then there is a unique homomorphism $\Phi : \ast_\alpha G_\alpha \to H$ so that the diagram below is commutative:

$$\begin{tikzcd}
\ast_\alpha G_\alpha \arrow[r, "\Phi"] & H \\
G_\beta \arrow[u, hook, "i_\beta"] \arrow[ur, "\varphi_\beta"'] &
\end{tikzcd}$$

$i_\beta : G_\beta \to \ast_\alpha G_\alpha$ is a monomorphism.

Theorem: There is a unique (up to isomorphism) group satisfying the property $\mathcal{P}$.

Idea of the proof: Elements of $\ast_\alpha G_\alpha$ are finite words of the form

$$g_1 g_2 g_3 \cdots g_n, \text{ where } g_i \in G_{\alpha_i},\ i=1,\dots,n.$$

Group operation: If $g_1 g_2 \cdots g_n$ and $g_1' g_2' \cdots g_m'$ are two words then

$$(g_1 g_2 \cdots g_n)\cdot(g_1' g_2' \cdots g_m') = g_1 g_2 \cdots g_n g_1' g_2' \cdots g_m'$$

Inverse: $(g_1 g_2 \cdots g_n)^{-1} = g_n^{-1} \cdots g_2^{-1} g_1^{-1}$ so that

$$(g_1 g_2 \cdots g_n)(g_n^{-1} \cdots g_2^{-1} g_1^{-1}) = g_1 g_2 \cdots \overset{e \in G_{\alpha_n}}{\overset{\|}{(g_n g_n^{-1})}} \cdots g_2^{-1} g_1^{-1}$$

$$= g_1 g_2 \cdots (g_{n-1} g_{n-1}^{-1}) \cdots g_2^{-1} g_1^{-1}$$

$$\vdots$$

$$= g_1 g_1^{-1}$$

$$= e$$

If we chose each G_α to be the cyclic group $(\mathbb{Z}, +)$ then the group $\underset{\alpha}{*} G_\alpha$ is called the free group on $|\Lambda|$ letters.

In particular, $|\Lambda| = n$ then we obtain

$\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-copies}}$ the free group on n-letters.

<u>Notation</u>: $F_n \doteq \mathbb{Z} * \cdots * \mathbb{Z}$

<u>Examples</u> 1) $F_2 = \mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle$

$$= \langle a, b \mid - \rangle$$

Elements of F_2 are words on the alphabeth a, a^{-1}, b, b^{-1}. For example,

$$ab,\ a^2 b,\ abab,\ aba^{-1},\ ab^2a^{-1}bab^{-1}a^2b^3, \ldots$$

$$abb^{-1}a^3 = aea^3 = aa^3 = a^4$$

<u>Definition</u>: Suppose that G is a group on the alphabeth $\{g_\lambda\}$ and let $R_\gamma \in G$, for each $\gamma \in R$, the set of relations.

Let N denote the smallest normal subgroup of G containing each R_γ, $\gamma \in R$. Then the quotient group G/N is called the group with generators $\{g_\lambda\}$ and relations $\{R_\alpha\}$.

$$G/N = \langle g_\lambda \mid R_\gamma \rangle$$

<u>Example:</u> $\mathbb{Z}_2 = \langle a \mid a^2 \rangle$, $\mathbb{Z}_n = \langle a \mid a^n \rangle$

$\mathbb{Z} = \langle a \mid - \rangle$.

$F_2 = \langle a, b \mid - \rangle$.

$\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2, b^3 \rangle \doteq F_2/N$

$N =$ normal closure of the set $\{a^2, b^3\}$.

$\mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL(2, \mathbb{Z})$

$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

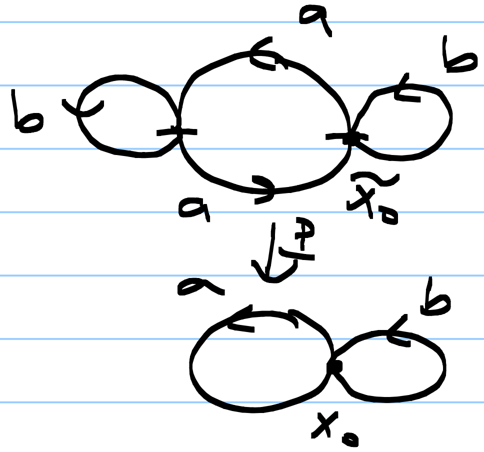
<u>Example:</u> $\mathbb{Z} \times \mathbb{Z}$ free abelian group of rank 2.

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle = F_2/F_2'$, where

$F_2' = [F_2, F_2]$ is the commutator subgroup.

In general $F_n/F_n' \simeq \mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$, the free abelian group of rank n.

Example: F_2 contains F_3 as a normal subgroup of index 2



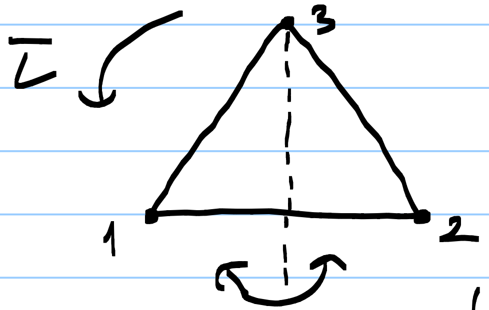
$F_2 = \langle a, b \mid - \rangle$

$H = \langle a^2, b, aba^{-1} \rangle$ the subgroup generated by a^2, b and aba^{-1}.

Claim: H is a normal subgroup of F_2 of index 2 isomorphic to F_3.

This will be proved using the theory of covering spaces.

Example: $S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \sigma\tau\sigma\tau \rangle$



τ = Counter clockwise $2\pi/3$ radian rotation

σ : reflection

$\sigma = (1\,2)$, $\tau = (1\,2\,3)$

Video 26

Theorem (Seifert, Van Kampen)

Let X be a topological space and U, V path connected open subsets of X so that $U \cap V$ is a nonempty path connected subset of X with $X = U \cup V$. Then the homomorphism

$$\Phi : \pi_1(U) * \pi_1(V) \longrightarrow \pi_1(X), \text{ where}$$

$\Phi(g) = i_{U\#}(g)$ $\forall\, g \in \pi_1(U)$ and

$\Phi(g) = i_{V\#}(g)$ $\forall\, g \in \pi_1(V)$, is surjective.

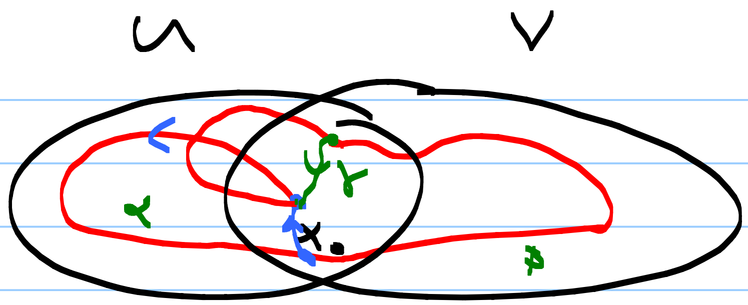
Moreover, the kernel N of Φ is generated by all elements of the form

$$(i_{U\#} \circ j_{U\#}(\omega))(i_{V\#} \circ j_{V\#}(\omega^{-1})), \text{ where } \omega \in \pi_1(U \cap V).$$

Notation: $\pi_1(X) \simeq \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "i_{U\#}"] & \\
\pi_1(U \cap V) \arrow[ur, "j_{U\#}"] \arrow[dr, "j_{V\#}"'] & & \pi_1(U \cup V) = \pi_1(X) \\
 & \pi_1(V) \arrow[ur, "i_{V\#}"'] &
\end{tikzcd}$$

Idea:



$\alpha\beta\gamma$

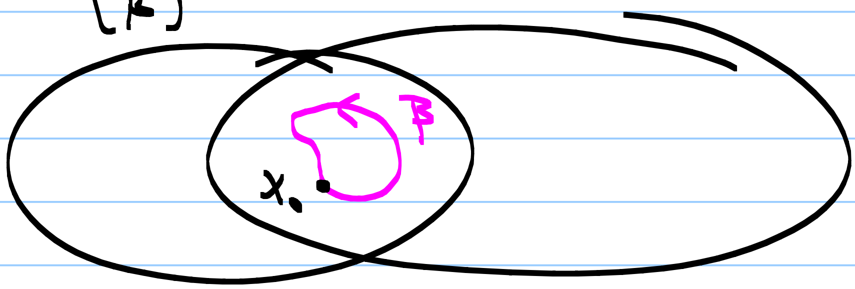
$\alpha, \gamma \in \pi_1(U, x_0)$, $\beta \in \pi_1(V, x_0)$.

$\pi_1(U, x_0) \xrightarrow{i_{U\#}} \pi_1(X, x_0)$ $\qquad i_U : U \hookrightarrow X$

$\pi_1(V, x_0) \xrightarrow[i_{V\#}]{} \pi_1(X, x_0)$ $\qquad i_V : V \hookrightarrow X$

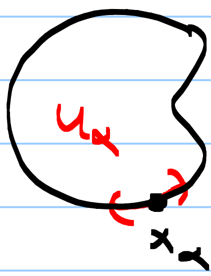
$\pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$

$[\beta]$ $\qquad [\beta]$



Examples: 1) $X = \bigvee_\alpha X_\alpha$, where each X_α is path connected and $x_\alpha \in X_\alpha$ so that there is some open subset $x_\alpha \in U_\alpha \subseteq X_\alpha$ which deformation retracts onto $\{x_\alpha\}$.

X_α



$X = \bigvee_\alpha X_\alpha \doteq \dot{\bigcup_\alpha} X_\alpha \Big/ x_\alpha \sim x_\beta$

$U = \bigvee_\alpha U_\alpha$ deformation retracts onto $\{x_0\}$, where $x_0 = [x_\alpha]$.

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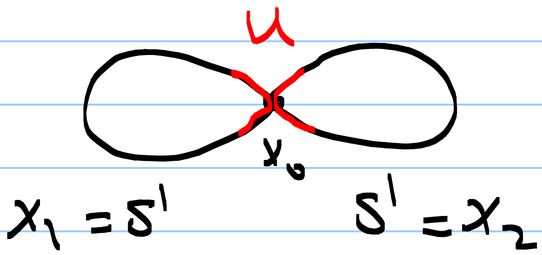
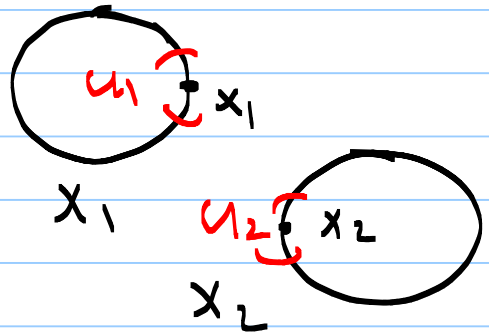
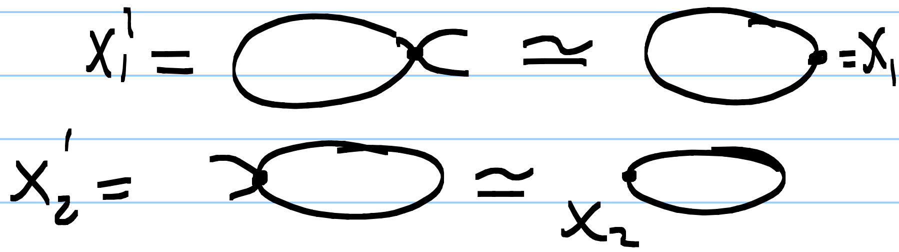
$$\pi_1(X) \doteq \underset{\alpha}{*}\, \pi_1(X_\alpha)$$

$X'_\alpha = X_\alpha \cup U$ open subset of X, which deformation retracts onto X_α.

<u>Example:</u> $S^1 \vee S^1$





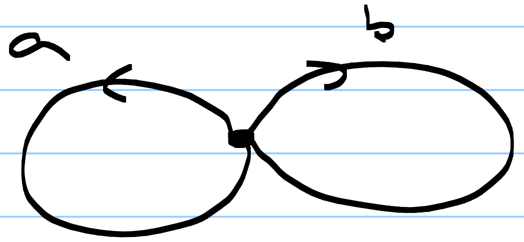


$$\pi_1(S^1 \vee S^1) \simeq \pi_1(X_1') * \pi_1(X_2') \simeq \pi_1(S^1) * \pi_1(S^1)$$

$\pi_1(X_1' \cap X_2') = (e)$, $\quad \pi_1(S^1) \cong \mathbb{Z}$, $\quad \pi_1(S^1) \cong \mathbb{Z}$

$$\simeq \mathbb{Z} * \mathbb{Z}$$

$$= F_2 = \langle a, b \mid - \rangle$$



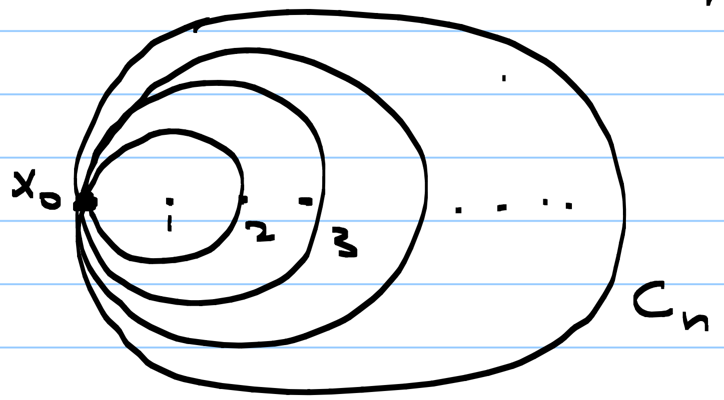
$a, b, a^2 b, a^{-1} b a^3, \ldots$

Similarly, $\pi_1(\bigvee_n S^1) \simeq \underset{n}{*}\, \pi_1(S^1) = \underset{n}{*}\, \mathbb{Z} = F_n$.

In general if $X = \bigvee_{\alpha \in \Lambda} S^1_\alpha$, then, $\pi_1(X)$ is free group on the Λ.

$\pi_1(X) = \langle g_\alpha, \alpha \in \Lambda \mid - \rangle$.

$\Lambda = \mathbb{N} = \{1, 2, 3, \dots\}$, $\pi_1(\bigvee_n S^1_n) = \underset{n\in\mathbb{N}}{*} \mathbb{Z} = F_\infty$



$X \subset \mathbb{R}^2$

Circles of radius n

$C_n : \{(x,y) \in \mathbb{R}^2 \mid (x-n)^2 + y^2 = n^2\}$

$$\begin{tikzcd} \coprod_n S^1_n \arrow[r, "\varphi"] \arrow[d] & X \\ \coprod_n S^1_n / \sim = \bigvee_n S^1_n \arrow[ur, dotted, "\tilde{\varphi}"] & \end{tikzcd}$$

$\varphi|_{S^1_n} : S^1_n \longrightarrow C_n$ homeomorphism

$\varphi(x_0^n) = x_0$, where $x_0^n \in S^1_n$ is a point.

$$\pi_1(\bigvee_n S^1_n) \xrightarrow[f_{n_0\#}]{} \pi_1(S^1_{n_0})$$

$f_{n_0} : \bigvee_n S^1_n \longrightarrow S^1_{n_0}$, $i_{n_0} : S^1_{n_0} \longrightarrow \bigvee_n S^1_n$

$f_{n_0} \circ i_{n_0} : S^1_{n_0} \longrightarrow S^1_{n_0}$, $f_{n_0} \circ i_{n_0} = \mathrm{Id}_{S^1_{n_0}}$

$$\begin{tikzcd} \pi_1(S^1_{n_0}) \arrow[r, "i_{n_0\#}"] & \pi_1(\bigvee_n S^1_n) \arrow[r, "f_{n_0\#}"] & \pi_1(S^1_{n_0}) \\ \mathbb{Z} \arrow[u, phantom, "\cong"] \arrow[rr, "\mathrm{id}"'] & & \mathbb{Z} \arrow[u, phantom, "\cong"] \end{tikzcd}$$

$f_n \circ i_m : S^1 \longrightarrow S^1$ is constant, if $m \neq n$.

Thus $(f_n \circ i_m)_\# : \mathbb{Z} \longrightarrow \mathbb{Z}$ is the trivial homomorphism if $m \neq n$.

$$\begin{array}{ccccc} \pi_1(S^1) & \xrightarrow{i_{m\#}} & \pi_1(\bigvee_n S^1) & \xrightarrow{f_{n\#}} & \pi_1(S^1) \\ \| & & & & \| \\ \mathbb{Z} & & & & \mathbb{Z} \end{array}$$

If $m_1, m_2, \ldots, m_k$ are different integers then

$$\pi_1\left(\bigvee_{i=1}^{k} S^1_{m_i}\right) \xrightarrow{i_\#} \pi_1\left(\bigvee_n S^1\right) \xrightarrow{P} \pi_1\left(\bigvee_{i=1}^{k} S_{m_i}\right)$$

i is the inclusion map and P is the map which identity on each $S^1_{m_i}$ and contraction on all other S^1_n's.

Since $P \circ i : S^1_{m_1} \vee \cdots \vee S^1_{m_k} \longrightarrow S^1_{m_1} \vee \cdots \vee S^1_{m_k}$ is the identity map and thus the homomorphism

$$(P \circ i)_\# : \pi_1\left(\bigvee_{i=1}^{k} S_{m_i}\right) \longrightarrow \pi_1\left(\bigvee_{i=1}^{k} S_{m_i}\right) \text{ is}$$

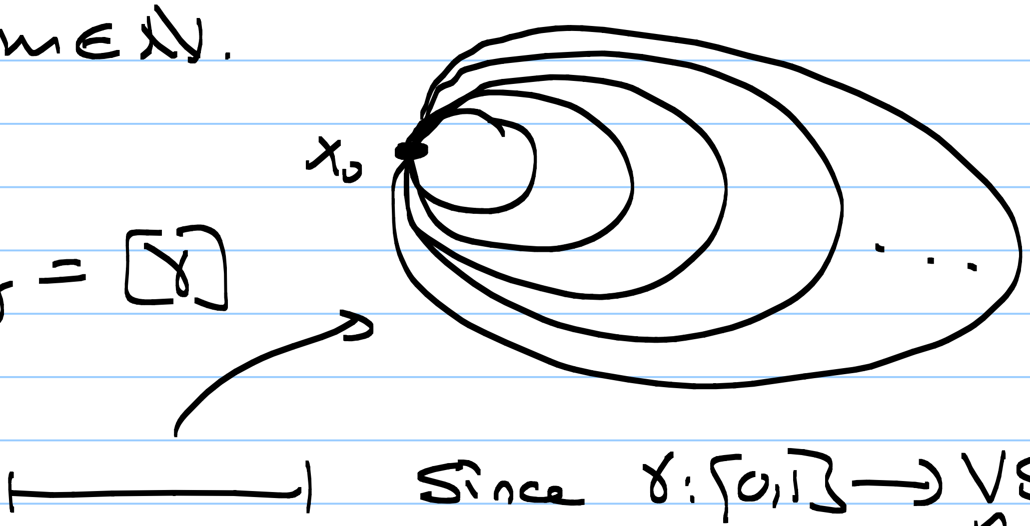
is identity. Hence $F_k \simeq i_\#\left(\pi_1\left(\bigvee_{i=1}^{k} S_{m_i}\right)\right)$ is a subgroup of $\pi_1(\bigvee_n S^1_n)$ $(\simeq F_\infty)$

Claim: Any element $g \in \pi_1(\bigvee_n S^1_n)$ maps to only finitely many non-trivial elements under projections:

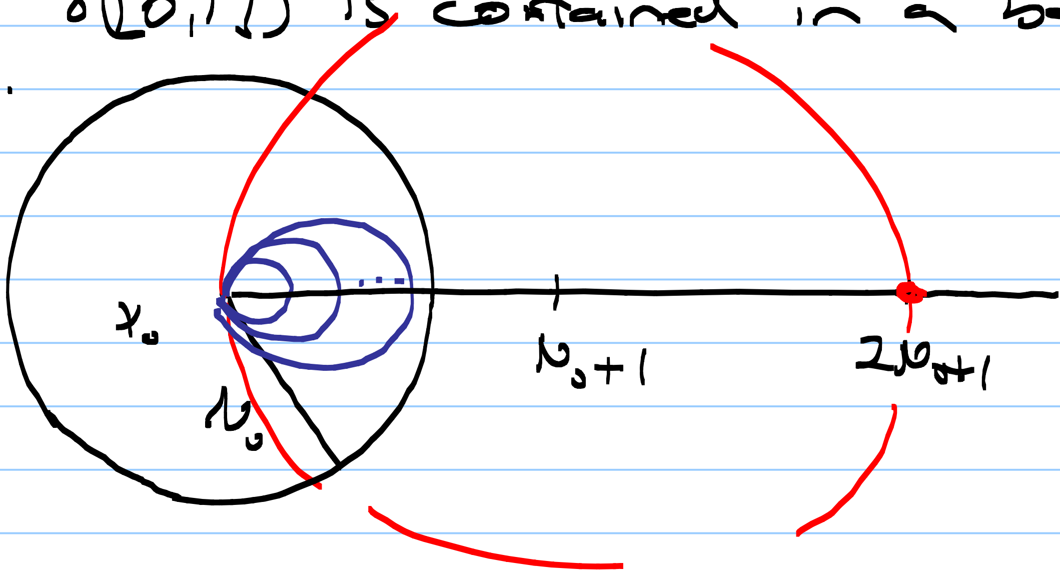
$$f_{m_\#}: \pi_1(\bigvee_n S^1_n) \longrightarrow \pi_1(S^1_m) \simeq \mathbb{Z}$$

$f_{m_\#}(g) = 0$ for all but finitely many $m \in \mathbb{N}$.

$g = [\gamma]$



Since $\gamma : [0,1] \longrightarrow \bigvee_n S^1 \subseteq \mathbb{R}^2$ is continuous and $[0,1]$ is compact $\gamma([0,1])$ is contained in a ball $B(0, N_0)$.



$(\gamma[0,1])$ does not contain $2n+1$ of $n \geq N_0$

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$[0,1] \xrightarrow{\gamma} \bigvee_n S^1_n \longrightarrow S^1_m$ is not onto if $m \geq n_0$ and thus on the π_1-level

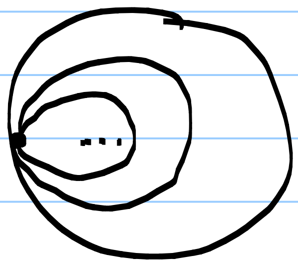
$$\begin{array}{ccc} \pi_1(\bigvee_n S^1_n) & \longrightarrow & \pi_1(S^1_m) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\cong$} \\ [g] & \longmapsto & 0 \in \mathbb{Z} \end{array}$$

Hence, g is a word in $\alpha_1^{\mp}, \dots, \alpha_{n_0}^{\mp}$.

$$g = \alpha_1^{3}\,\alpha_2^{-5}\,\alpha_4^{2}\,\alpha_5^{7} \cdots \alpha_{n_0}^{-4}$$

Conclusion: $\pi_1(\bigvee_n S^1_n)$ is free group on $\mathbb{N}$.

Example: $Y = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{(x-\frac{1}{n^2})^2 + y^2 = \frac{1}{n^4}}_{C_n}, n \in \mathbb{N}\}$



C_n

$$\sum_{n=1}^{\infty} \frac{2\pi}{n^2} = \frac{\pi^2}{6} \cdot 2\pi = \frac{\pi^3}{3}$$

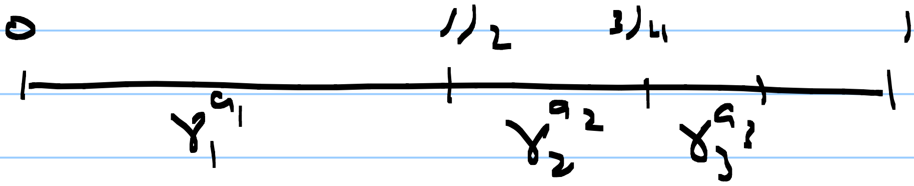
$\pi_1(Y)$ is uncountable!

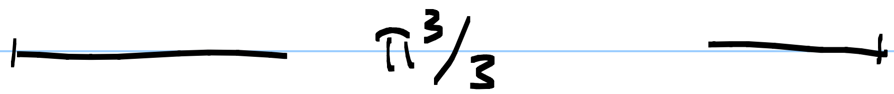
To show that $\pi_1(Y)$ is uncountable we find an injective map from $[0,1]$ to $\pi_1(Y)$:

$x \in [0,1]$, $x = 0.\underset{1.}{3}\underset{2.}{5}\underset{3.}{0}8216\underset{n.}{9}0032 = 0.a_1a_2\cdots$

Map x to the loops that goes around C_n as many times as the n^{th} decimal.
If $[\gamma_i]$ represents the loop that goes around the i^{th} circle once then x corresponds to

$$x \longmapsto \gamma_1^{a_1}\gamma_2^{a_2}\cdots\cdots\gamma_n^{a_n}\cdots\cdots$$





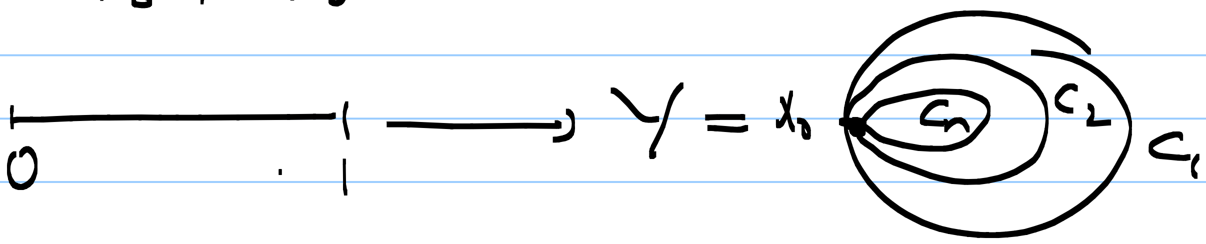
$\Rightarrow$ continuous loop.

So we get a map $[0,1] \longrightarrow \pi_1(Y)$

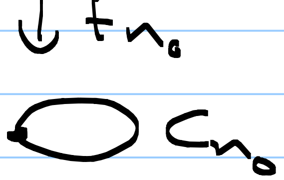
If $x, y \in [0,1]$ and $x \neq y$ then there is some $n_0 \in \mathbb{N}$ so that
$x = 0.a_1a_2\cdots a_{n_0}a_{n_0+1}\cdots$, $y = 0.b_1b_2\cdots b_{n_0}b_{n_0+1}\cdots$

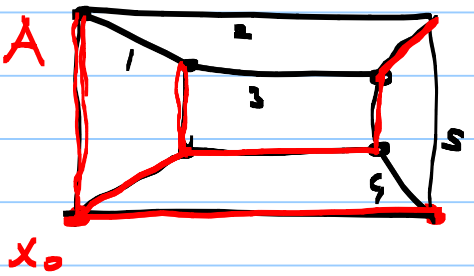
and $a_i = b_i$ for $i < n_0$ and

$a_{n_0} \neq b_{n_0}$.



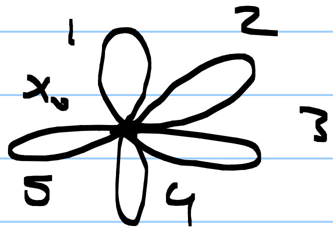
$$[0,1] \longrightarrow \pi_1(Y) \xrightarrow{(f_{n_0})_\#} \pi_1(C_{n_0}) \simeq \mathbb{Z}$$
$$x \longmapsto a_{n_0}$$
$$y \longmapsto b_{n_0}$$
$a_{n_0} \neq b_{n_0}$



Example: X:  $A \subseteq X$ is contractible (to the point x_0)

X 1-dim'l CW-complex
$A \subseteq X$ subcomplex

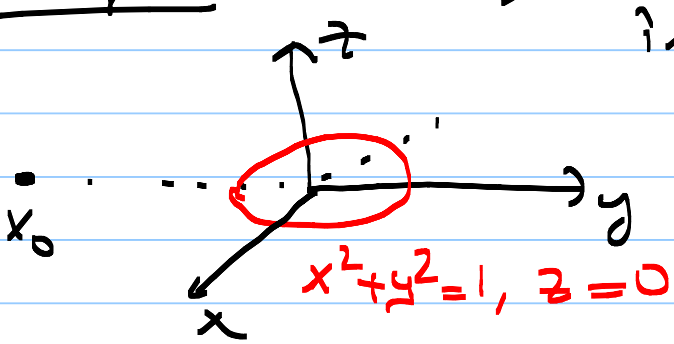
Since A is contractible X is homotopy equivalent to X/A.



Hence, X/A is homeomorphic to $\bigvee_5 S^1$.

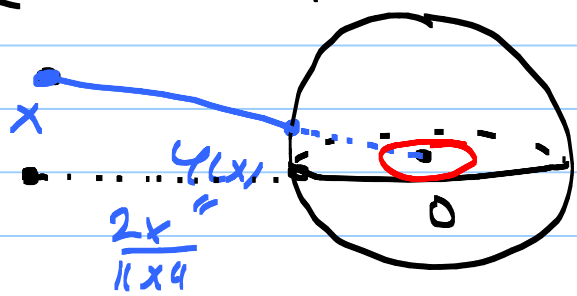
So, $\pi_1(X) \simeq \pi_1(X/A) \cong \pi_1(\bigvee_5 S^1) \cong F_5$

Example: $\mathbb{R}^3 \setminus S^1$, $S^1 \subseteq \mathbb{R}^3$ the unit circle in $\mathbb{R}^2 \times \{0\}$.



$\pi_1(\mathbb{R}^3 \setminus S^1) = ?$

$(\mathbb{R}^3 \setminus S^1)$ deformation retracts onto $D^3 \setminus S^1$.



$D^3 = B[0,2]$

So, $\pi_1(\mathbb{R}^3 \setminus S^1) \simeq \pi_1(D^3 \setminus S^1)$.

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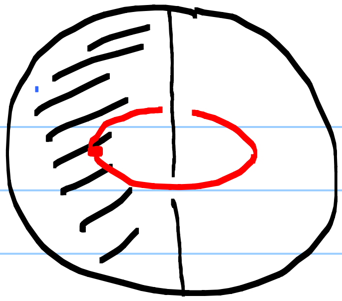
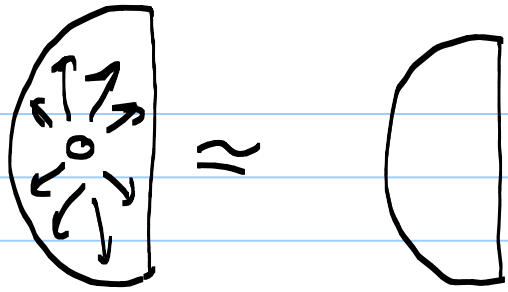
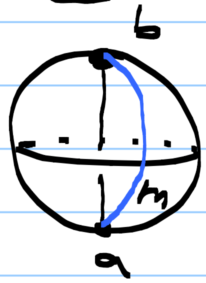


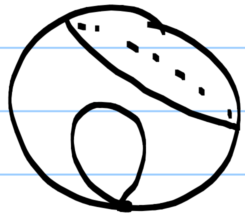


Hence, $D^3 \setminus S^1$ deformation retracts onto $S^2 \cup [a,b]$



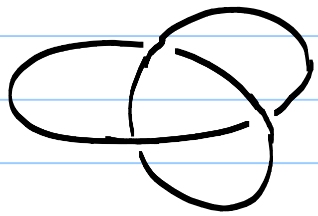
$\pi_1(D^3 \setminus S^1) \simeq \pi_1(S^2 \cup [a,b])$

$S^2 \cup [a,b] / m = S^2 \vee S^1$



$$\pi_1(S^2 \vee S^1) = \underbrace{\pi_1(S^2)}_{\overset{\shortparallel}{(e)}} * \underbrace{\pi_1(S^1)}_{\overset{\cong}{\mathbb{Z}}} \simeq \mathbb{Z}$$

Remark: $K \subseteq S^3$ knot (embedded circle)



$\pi_1(S^3 \setminus K)$ knot group

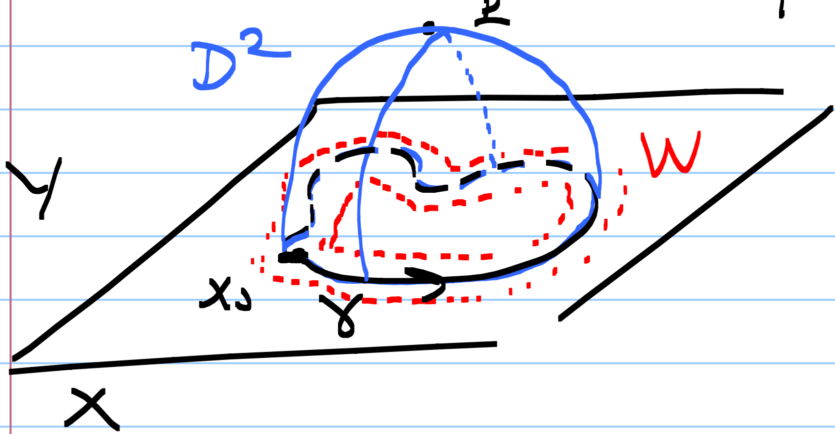
Proposition: Let (X, x_0) be a based topological space and $\gamma : [0,1] \longrightarrow X$ be a loop representing the element $[\gamma]$ in $\pi_1(X, x_0)$. Let Y be the attaching space defined by

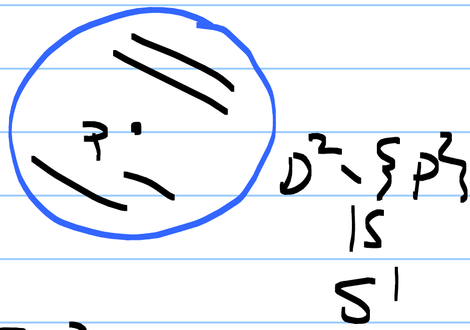
$$Y \doteq X \,\dot\cup\, D^2 / \gamma(t) \sim (\cos 2\pi t, \sin 2\pi t), \; t \in [0,1]$$

where $(\cos 2\pi t, \sin 2\pi t)$ parametrizes the boundary $S^1 = \partial D^2$. Then $\pi_1(Y, x_0)$ is isomorphic to

$$\pi_1(Y, x_0) = \pi_1(X, x_0) / N_{[\gamma]}, \quad \text{where } N_{[\gamma]} \triangleleft \pi_1(X, x_0)$$

is the normal closure of $[\gamma]$ in $\pi_1(X, x_0)$.





__Proof__: $Y = U \cup V$, $U = Y \setminus \{p\}$ and

V is a open neighborhood of D^2 in Y.

$V = D^2 \cup W$. We assume W deformation retracts onto γ. Hence, V deformation retracts to the point $\{p\}$, so it is contractable. Note that U deformation retracts onto X. Moreover, $U \cap V = V \setminus \{p\}$ and it deformation retracts onto W which deformation retracts onto γ.

Now Apply Zariski-van Kampen's Theorem to this setting

$$\begin{array}{ccccc} & & \pi_1(U) = \pi_1(X) & & \\ & \nearrow & & \searrow & \\ \pi_1(U \cap V) & & & & \pi_1(U \cup V) = \pi_1(Y) \\ \text{is } \mathbb{Z} = \langle [\gamma] \rangle & \searrow & & \nearrow & \\ & & \pi_1(V) & & \\ & & \text{"} (e) & & \end{array}$$

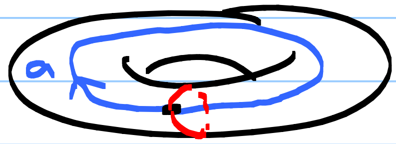
$$\text{So } \pi_1(Y) = \underbrace{\pi_1(U)}_{\pi_1(X)} * \underbrace{\pi_1(V)}_{\{e\}} \Big/ N_{[\gamma]}$$

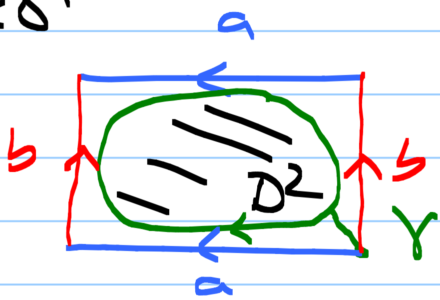
$$= \pi_1(X)/N_{[\gamma]} \qquad \Big(N_{[\gamma]} = \bigcap_{\substack{N \trianglelefteq G \\ [\gamma]\in N}} N \Big)$$

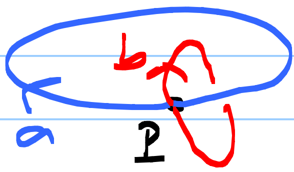
$$\pi_1(X) = \langle g_\alpha \mid r_i \rangle$$

$$\pi_1(Y) = \langle g_\alpha \mid r_i , [\gamma] \rangle$$

<u>Examples</u> 1) $T^2 = S^1 \times S^1$





X: 

$\simeq S^1 \vee S^1$, $\pi_1(X,p) = \langle a,b \mid - \rangle$

$Y = X \cup D^2$, $\pi_1(Y) = \langle a,b \mid [\gamma] \rangle$

$aba^{-1}b^{-1} = e$

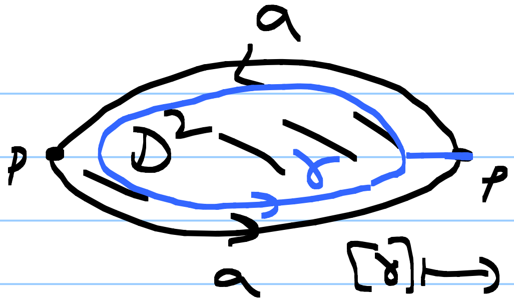
$ab = ba$

$= \langle a,b \mid aba^{-1}b^{-1} \rangle$

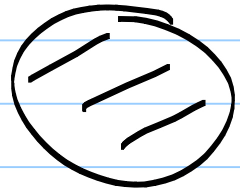
= the free abelian group of rank two

$$= \underbrace{\mathbb{Z}}_{\langle a\rangle} \oplus \underbrace{\mathbb{Z}}_{\langle b\rangle}$$

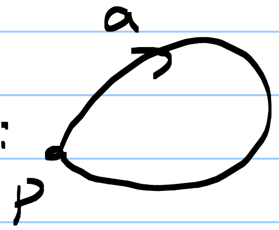
5) $\mathbb{RP}^2$



$[\gamma] \longmapsto [a]^2$



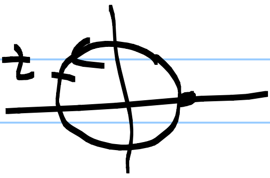
D^2

X: 

$\partial D^2 = S^1 \longrightarrow S^1, \; z \longmapsto z^2$

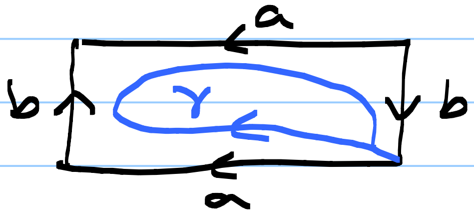
$\overset{\shortparallel}{\partial D^2} \qquad \overset{\shortparallel}{a \cup \{p\}}$

$\pi_1(X) = \langle a | - \rangle \; \mathbb{Z}$



$$\pi_1(\mathbb{RP}^2) = \pi_1(X \cup D^2) = \langle a \,|\, a^2 \rangle \simeq \mathbb{Z}_2.$$

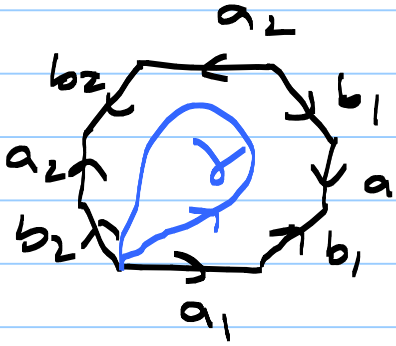
6) KB: Klein Bottle



$\gamma \longmapsto a b a^{-1} b$

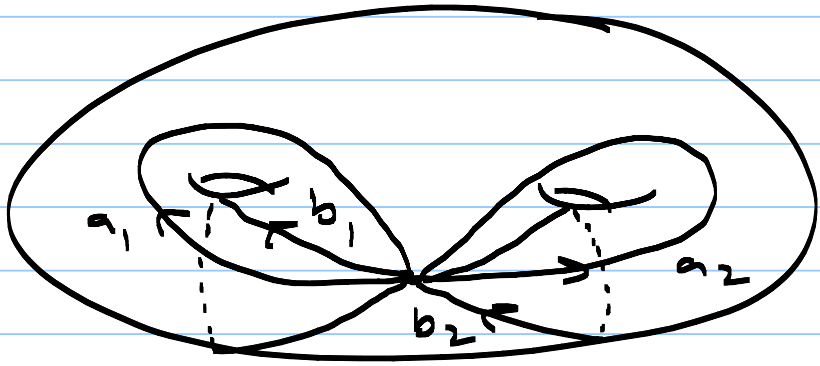
$$\pi_1(KB) = \langle a, b \,|\, a b a^{-1} b \rangle$$
$$= \langle a, b \,|\, a b a^{-1} = b^{-1} \rangle$$

7) $T^2 \# T^2$



$\gamma \longmapsto a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$
$= [a_1, b_1][a_2, b_2]$

$$\pi_1(T^2 \# T^2) = \langle a_1, b_1, a_2, b_2 \,|\, [a_1, b_1][a_2, b_2] \rangle$$
$$\overset{\shortparallel}{\Sigma_2}$$



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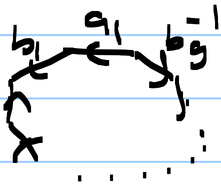
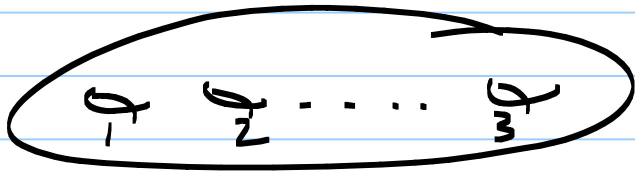
8) Exercise: $\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g\text{-copies}} = \Sigma_g$





$a_1 b_1 a_1^{-1} b_1^{-1} \cdots\cdots a_g b_g a_g^{-1} b_g^{-1}$

9) $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) * \pi_1(\mathbb{RP}^2)$
$= \mathbb{Z}/_2 * \mathbb{Z}/_2$

10) $X_n = e_0 \vee e_1 \vee e_2 \qquad S^1 = \partial D^2 \longrightarrow S^1 = e_0 \cup e^1$
$z \longmapsto z^n$

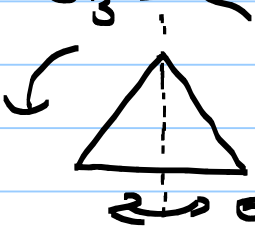
$\pi_1(X_n) = \langle a \mid a^n \rangle \simeq \mathbb{Z}/_{n\mathbb{Z}} \simeq \mathbb{Z}_n$

$\pi_1(X_2 \vee X_3) = \pi_1(X_2) * \pi_1(X_3) \simeq \mathbb{Z}/_2 * \mathbb{Z}/_3$
$\simeq PSL(2,\mathbb{Z})$

X_n :

11) $S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \underline{\sigma\tau\sigma = \tau^{-1}} \rangle$
$\sigma\tau\sigma\tau$

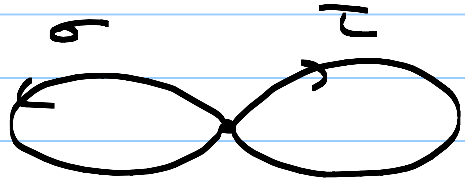
τ



σ : reflection

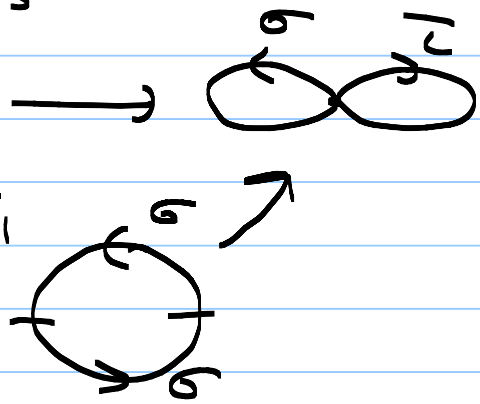
$\tau : \frac{2\pi}{3}$ radian rotation

$X = e^0 \vee e^1_1 \vee e^1_2 \vee e^2_1 \cup e^2_2 \vee e^2_3$

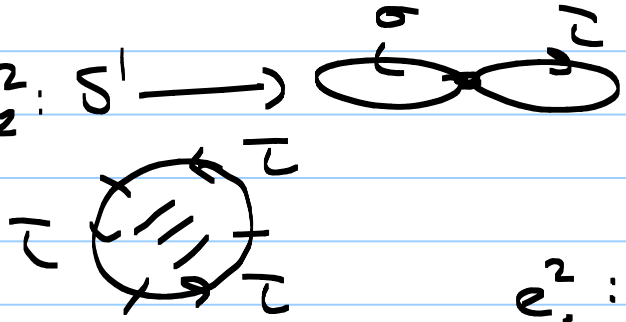


$e^2_1 : S^1 \longrightarrow$

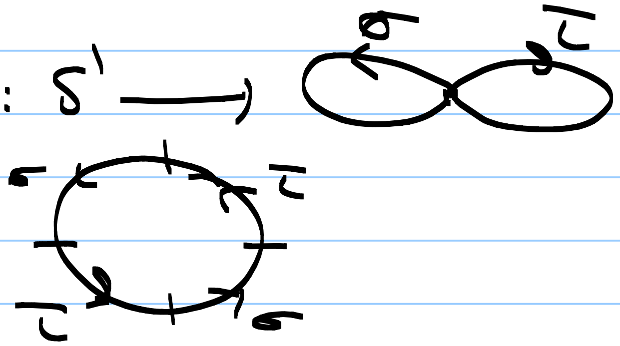
$\overset{\shortparallel}{\partial e^2_1}$



$e^2_2 : S^1 \longrightarrow$



$e^2_3 : S^1 \longrightarrow$



Exercise: Find a space X so that $\pi_1(X) = (\mathbb{Q}, +)$.

$\langle 1 \rangle = \mathbb{Z} \leadsto g_1 = 1 \in \mathbb{Q}$

$\langle 1/2 \rangle = \mathbb{Z}_{1/2} = \langle \ldots, -1/2, 0, 1/2, 1, 3/2, \ldots \rangle \leadsto g_2 = 1/2$

$\langle 1/n \rangle = \mathbb{Z}_{1/n} = \langle \ldots, -\frac{1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \rangle \leadsto g_n = 1/n$

$g_1 = g_2^2, \quad g_1 g_2 = g_2 g_1, \ldots$

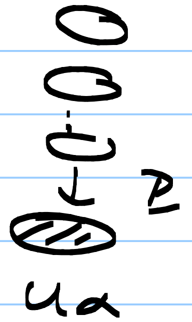
$\mathbb{Q} = \lim_{n \to \infty} \mathbb{Z}_{1/n}$

$$\begin{tikzcd}
 & \mathbb{Z}_{1/2} \arrow[dl, hook'] & \\
\mathbb{Z}_{1/6} & & \mathbb{Z} \arrow[ul, hook'] \arrow[dl, hook] \\
 & \mathbb{Z}_{1/3} \arrow[ul, hook] &
\end{tikzcd}$$

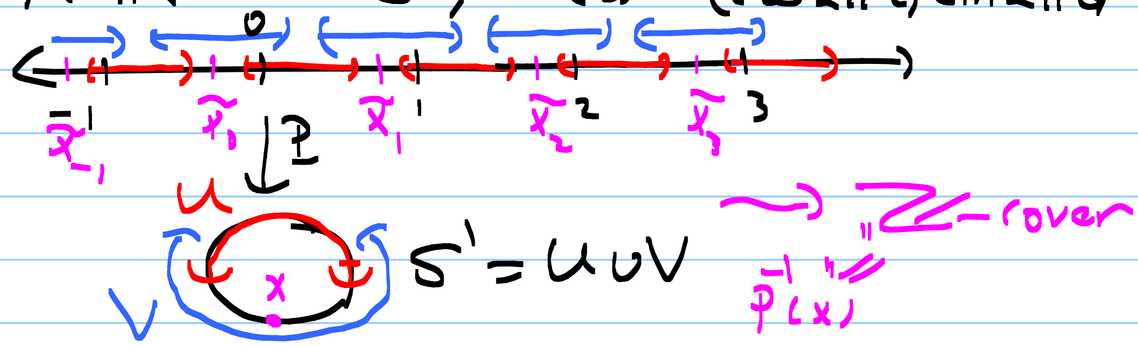
$$\begin{tikzcd}
 & 2g_2 \arrow[dl] & \\
6g_6 & & g_1 \arrow[ul] \arrow[dl] \\
 & 3g_3 \arrow[ul] &
\end{tikzcd}$$

Covering Spaces:

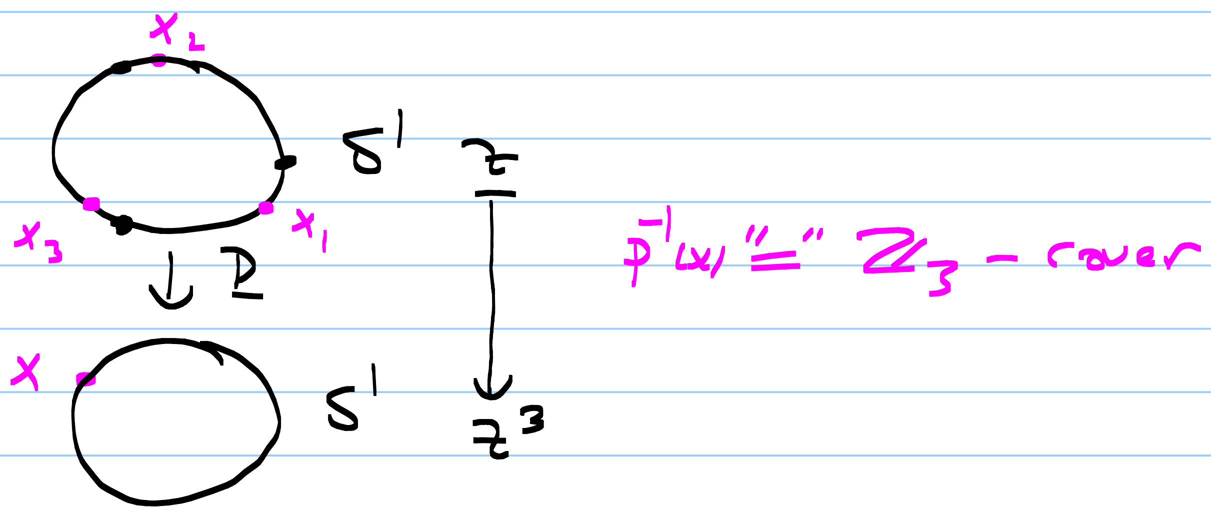
Definition: A covering space of a space X is a map $p:\tilde{X}\to X$, where $\tilde{X}$ is another space so that there is an open cover $\{U_\alpha\}_{\alpha\in A}$ of X, where each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U_α via p.



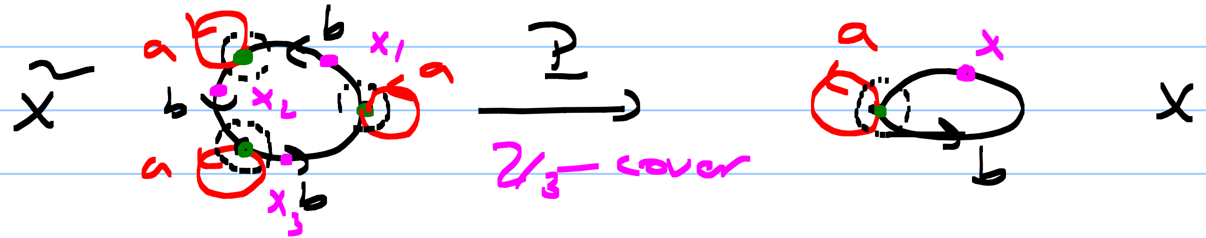
Example 1) $p:\mathbb{R}\longrightarrow S^1$, $p(t)=(\cos 2\pi t, \sin 2\pi t)$



2)



3)



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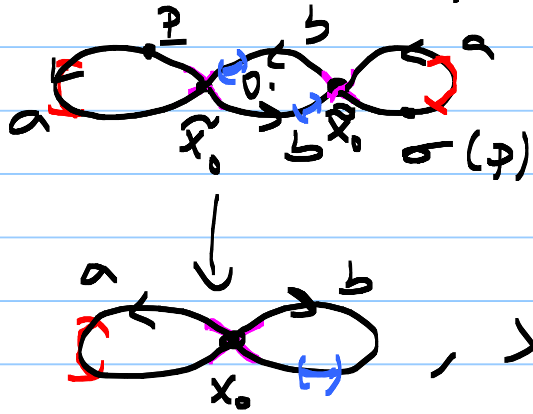
4) $p: S^n \longrightarrow \mathbb{RP}^n = S^n /_{p \sim -p}, \ p \in S^n$

$p = (x_1, \dots, x_{n+1}) \in S^n$

p is 2–1 map. $\quad p(p) = [p] = [x_1 : \cdots : x_{n+1}]$.

Double cover or equivalently a $\mathbb{Z}_2$-cover.

5) $\tilde{X}$



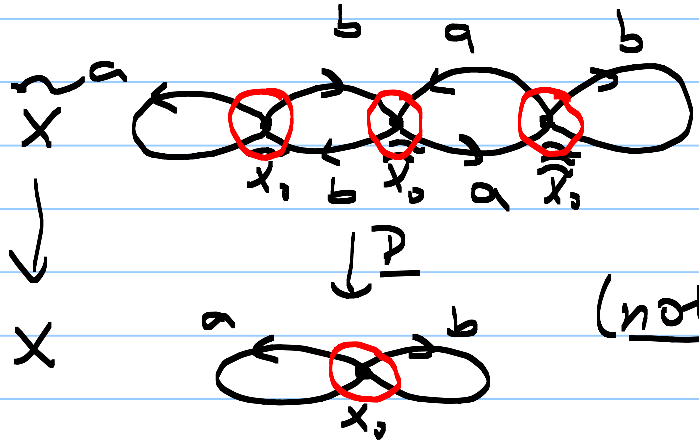
σ : π-radian rotation about 0.

X:

$X = \tilde{X} /_{p \sim \sigma(p)}, \ p \in X$

$\mathbb{Z}_2$-cover

6) $\tilde{X}$

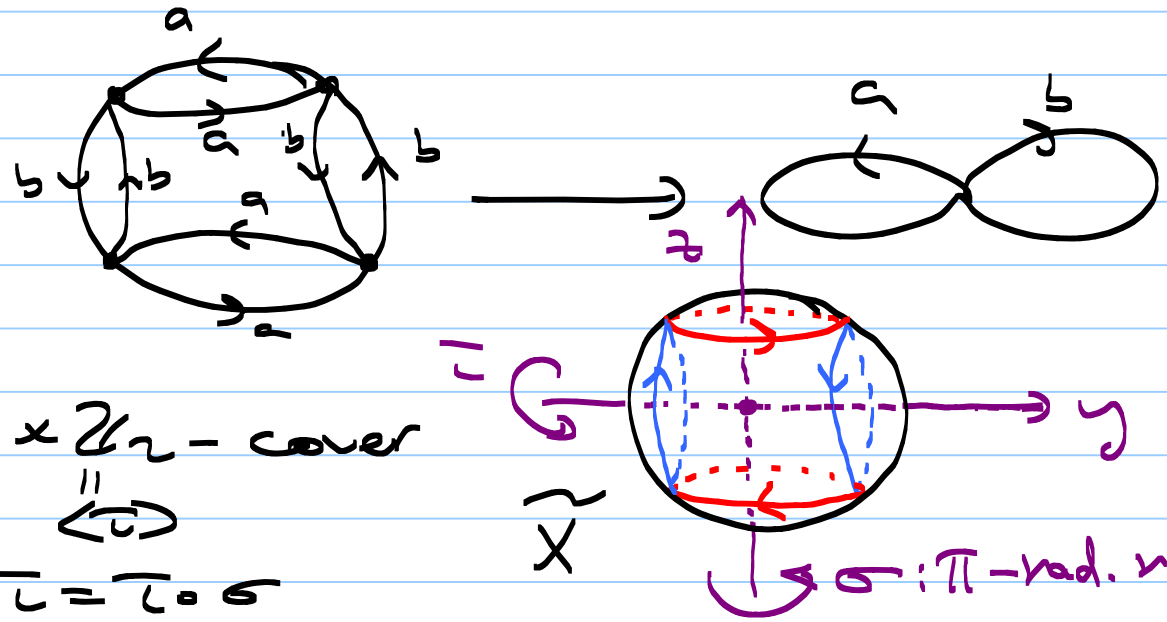


X

This is a 3-fold covering.

(<u>not</u> a regular cover)

7)



τ

$\tilde{X}$

σ : π-rad. rot.

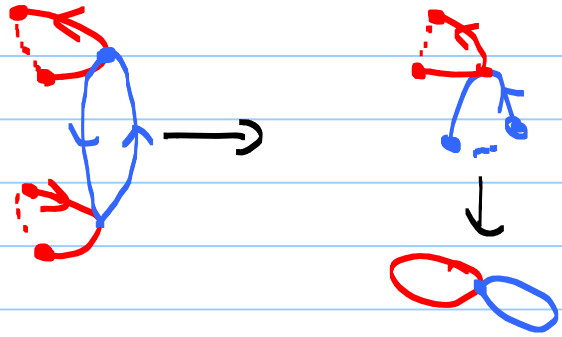
$\mathbb{Z}_2 \times \mathbb{Z}_2$ - cover

$\langle\sigma\rangle = \mathbb{Z}_2, \quad \langle\tau\rangle = \mathbb{Z}_2$

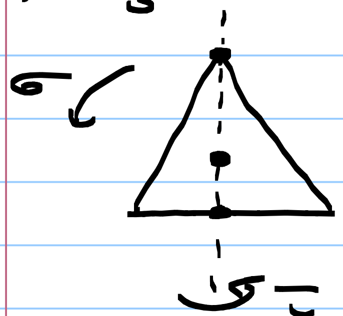
$\sigma \circ \tau = \tau \circ \sigma$

$\sigma(x,y,z) = (-x,-y,z)$ and $\tau(x,y,z) = (-x,y,-z)$ for all $(x,y,z) \in S^2$.

$\tilde{X} / \mathbb{Z}_2 \times \mathbb{Z}_2 = ?$



8) S_3 – covering



$S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, \tau\sigma\tau\sigma \rangle$

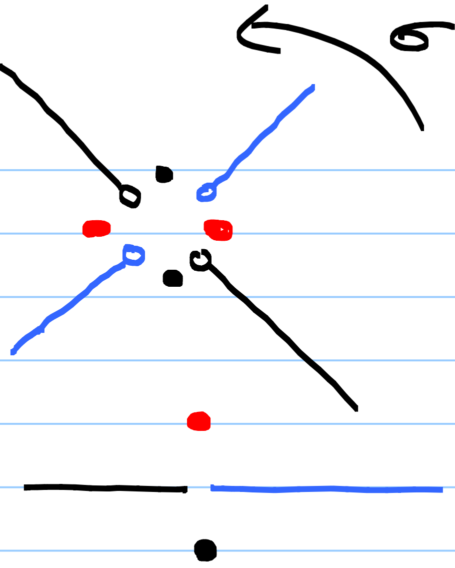
τ : reflection

σ : $2\pi/3$ rad. rotation

$F_7 \triangleleft F_2$ (normal subgroup)

of index 6 with $F_2 / F_7 \simeq S_3$.

9) $\tilde{X}$:



σ : π-radian rotation

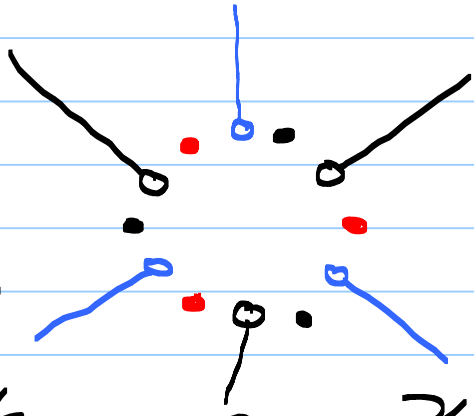
$\mathbb{Z}_2$-cover

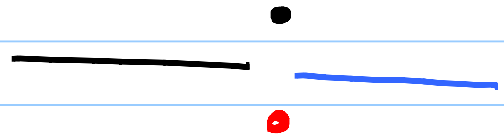
$\tilde{X}/\sigma \simeq X$

$\downarrow p$

X:

10)



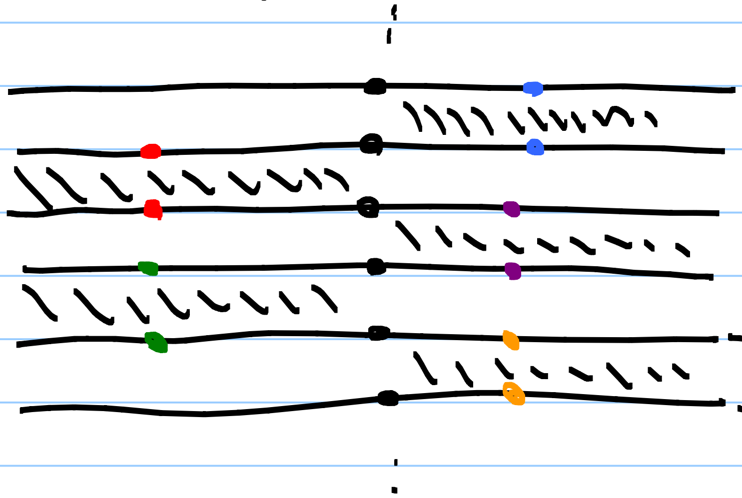


$: X$

τ : $2\pi/3$ rotation

$\tilde{X} \xrightarrow{\mathbb{Z}_3\text{-cover}} X$

11) $\tilde{X} = \mathbb{R} \times \mathbb{Z}/\sim$



$(x,n) \sim (x,n+1)$ if and only if either $(x>0$ and n is even$)$ or $(x<0$ and n is odd$)$.

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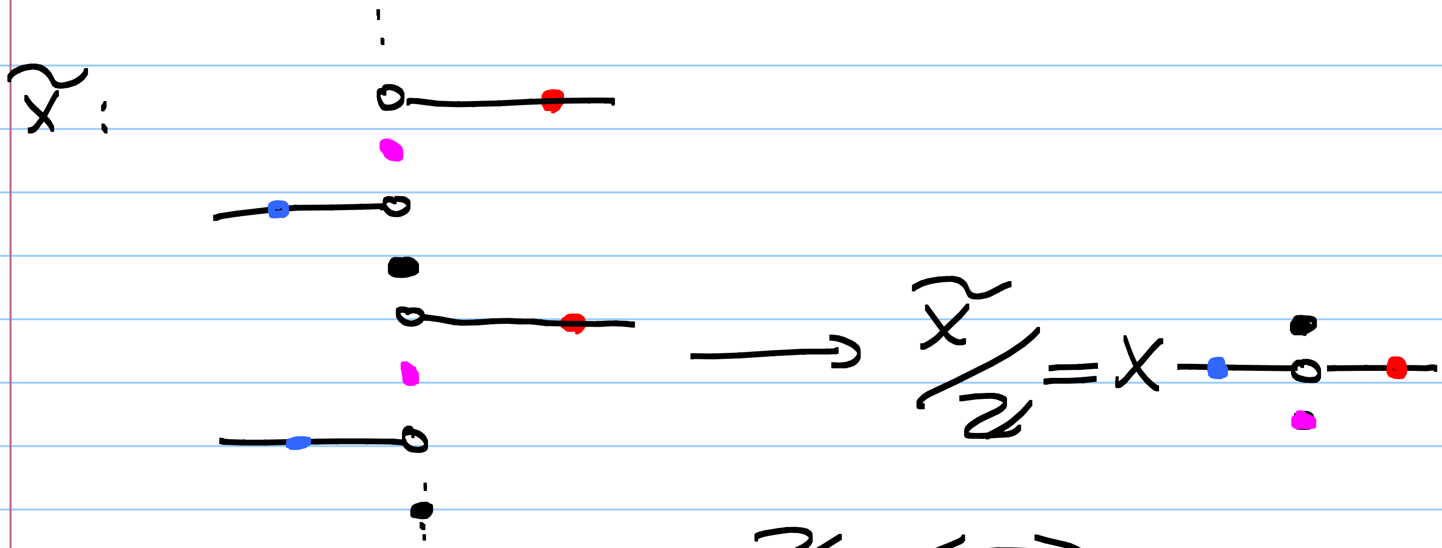
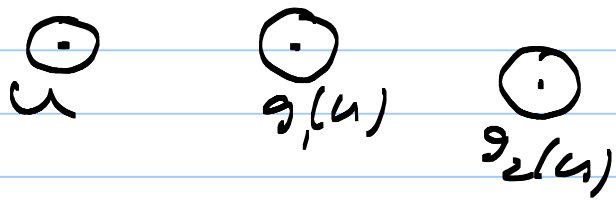


$\mathbb{Z} = \langle \sigma \rangle$

Regular $\mathbb{Z}$-covering.

Definition: A group G is said to act properly discontinuous on a space X if for every $g \in G$ and $x \in X$ there is an open set $x \in U \subseteq X$ so that
$$g_1(U) \cap g_2(U) = \emptyset \text{ whenever } g_1 \neq g_2.$$
$(G \leq \text{Homeo}(X))$



Proposition: If a group $G \leq \text{Homeo}(X)$ acts freely and properly discontinuous on X then the quotient map
$$P : X \longrightarrow X/G, \quad x \sim g(x),\ g \in G,\ x \in X$$
is a covering space.

Such covering spaces are called regular.

Lifting Properties:

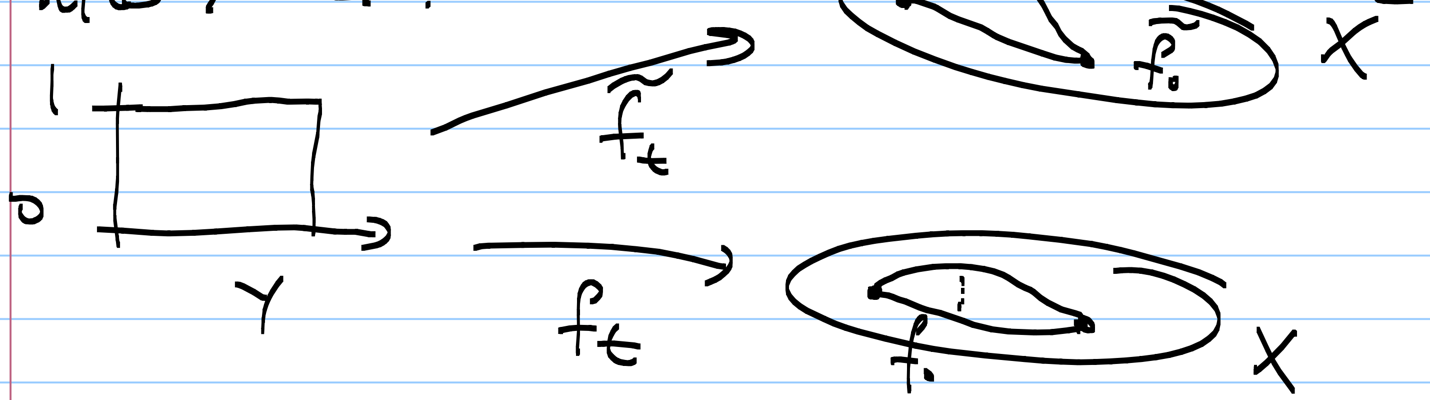
Let $p: \tilde{X} \to X$ be a covering space and $f: Y \to X$ is any map. A lifting of $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ so that $f = p \circ \tilde{f}$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \qquad f = p \circ \tilde{f}$$

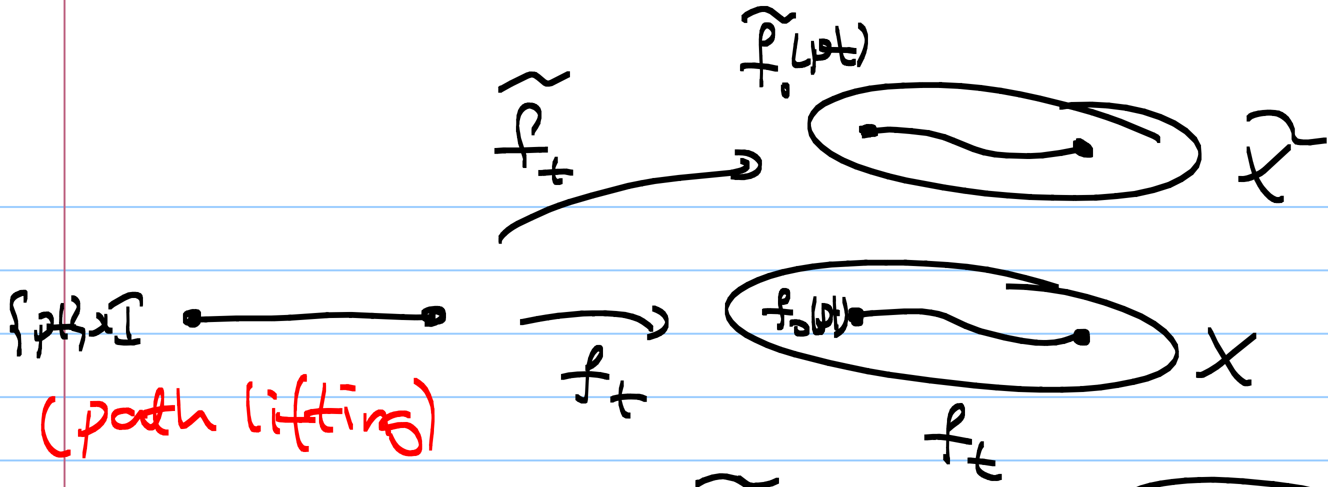
Proposition: (Homotopy lifting)

Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$ and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting $f_0: Y \to X$, then there is a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Proof: This is Part (c) of the proof that $\pi_1(S^1) \cong \mathbb{Z}$.

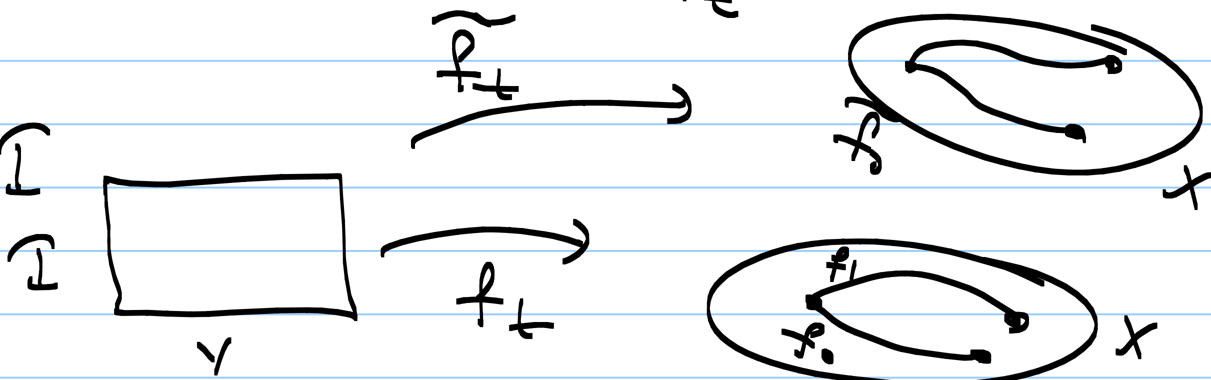


Special Cases: 1) $Y = \{pt\}$, $Y \times I = I$



(path lifting)

2) $Y = I$



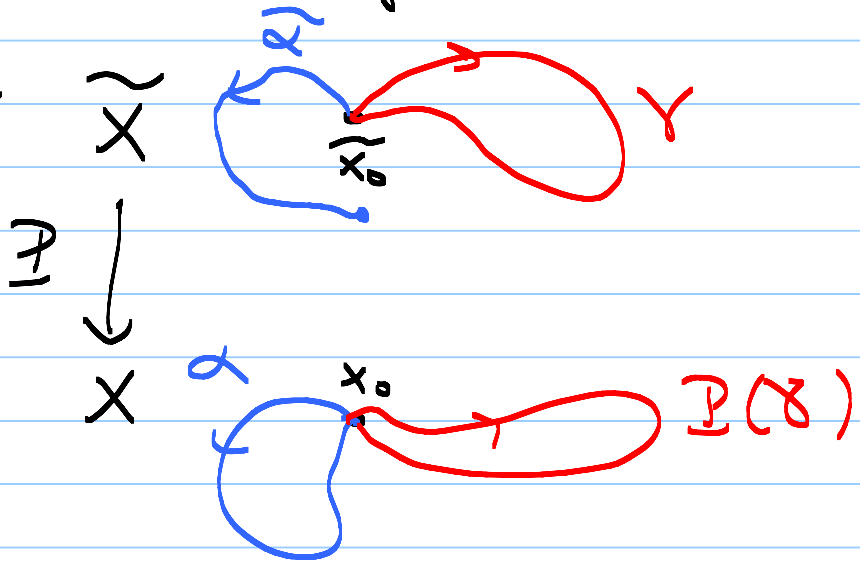
(Homotopy lifting)

Proposition: Let $p:(\tilde{X},\tilde{x}_0)\longrightarrow(X,x_0)$ be a covering space. Then the homomorphism

$$p_\#:\pi_1(\tilde{X},\tilde{x}_0)\longrightarrow\pi_1(X,x_0)$$ induced by p,

is injective. The image subgroup $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ consists of loops in X based at x_0, whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Proof:



Proof: Let $\tilde{f}_0: I\longrightarrow\tilde{X}$ be a loop based at $\tilde{x}_0$, which represents a class in the kernel of the homomorphism $p_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$.

To show that $p_{\#}$ is injective we must show $\tilde{f}_0$ is homotopic to the constant path at $\tilde{x}_0$.

Then $\tilde{f}_0$ is the unique lifting of $f_0 \doteq p \circ \tilde{f}_0 : I \to X$, a loop in X based at x_0. By assumption f_0 is homotopic to a constant. Hence, there is a homotopy f_t joining f_0 to the constant loop at x_0.

By the previous proposition there is a unique homotopy $\tilde{f}_t$ of $\tilde{f}_0$ to some $\tilde{f}_1$ so that

$$p \circ \tilde{f}_t = f_t \quad \text{for all } t.$$

In particular, $p \circ \tilde{f}_1(s) = f_1(s) = x_0$, for all $s \in [0,1]$. Hence, $\tilde{f}_1(s) \in p^{-1}(x_0)$, for all $s \in [0,1]$. Since $p^{-1}(x_0)$ is a discrete set and $\tilde{f}_1(0) = \tilde{x}_0$ we see that $\tilde{f}_1(s) = \tilde{x}_0$, for all $s \in [0,1]$.

In particular, $\tilde{f}_1$ is a constant loop homotopic to $\tilde{f}_0$. This shows that $p_{\#}$ is an injective group homomorphism

The proof of the second statement is left as an exercise. ●

If $p: \tilde{X} \to X$ is a covering then for any $x \in X$ the cardinality of $p^{-1}(x)$ is called the "number of sheets" of the covering above x.

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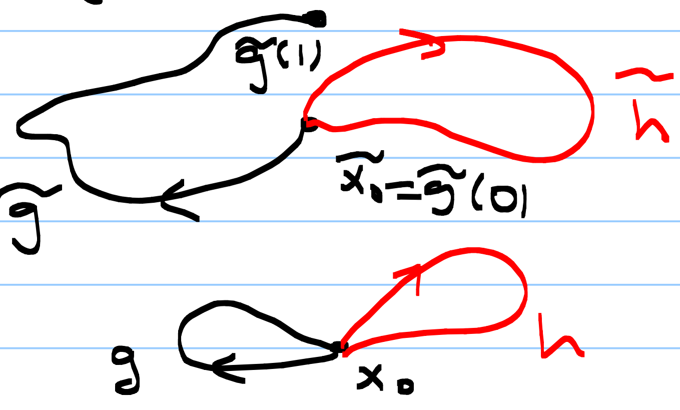
Proposition: The number of sheets $|p^{-1}(x_0)|$ of a covering $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ of path connected spaces is equal to the index of the subgroup $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$.

Proof: Let g be a loop at x_0 and $\tilde{g}$ be the lift to $\tilde{X}$ starting at $\tilde{x}_0$. If $[h]\in H=p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ and $\tilde{h}$ is lift of h then the lift $\tilde{h}\tilde{g}$ of hg starting at $\tilde{x}_0$ has the same end point with $\tilde{g}$.

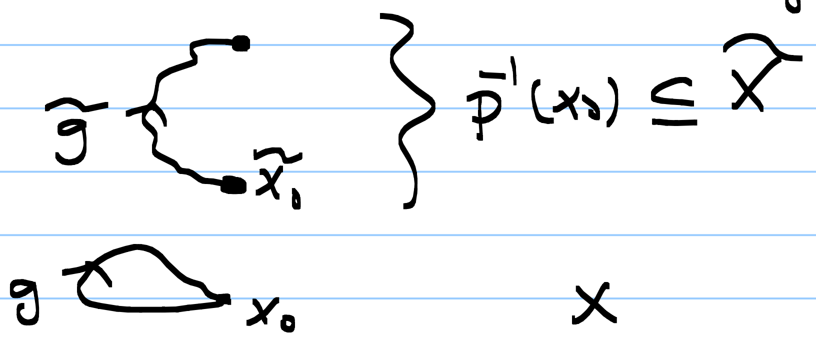


So we get a well defined function

$$\Phi:\{H[g]\mid [g]\in\pi_1(X,x_0)\}\longrightarrow p^{-1}\{x_0\}$$ by

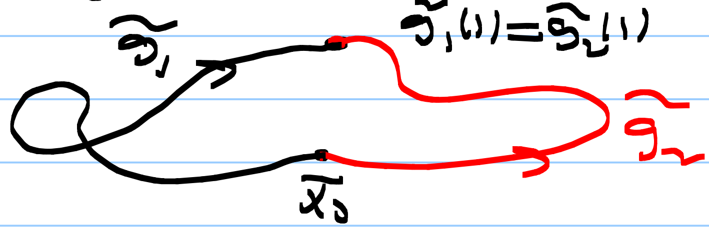
sending the coset $H[g]$ to the end point $\tilde{g}(1)$.

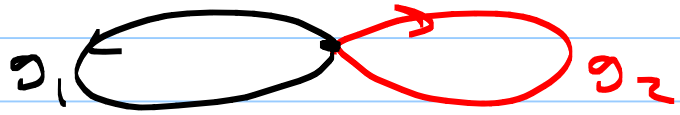
Since $\tilde{X}$ is connected Φ is surjective.



To show that Φ is injective assume that
$\Phi(H[g_1]) = \Phi(H[g_2])$, for some $[g_i] \in \pi_1(X, x_0)$, $i=1,2$.

Then if $\tilde{g}_1$ and $\tilde{g}_2$ are the lifts of g_1 and g_2 starting at $\tilde{x}_0$ then $\tilde{g}_1(1) = \tilde{g}_2(1)$.





Hence $\tilde{g}_1 \bar{\tilde{g}}_2$ is a loop at $\tilde{x}_0$. Hence, $[g_1 \bar{g}_2]$ is contained in the subgroup $H = p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$.

$[g_1 \bar{g}_2] = [g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.

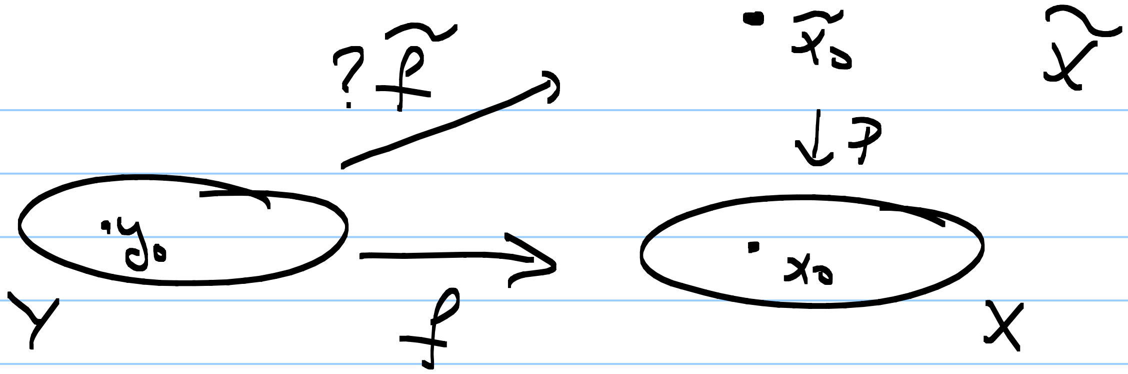
This finishes the proof. ∎

Proposition: (Lifting Criterion)

Let $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be a covering space, $f : (Y, y_0) \longrightarrow (X, x_0)$ a map, where $f(y_0) = x_0$ and Y is path connected and locally path connected. Then f has a lift

$$\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0) \text{ if and only if } f_{\#}(\pi_1(Y, y_0)) \subseteq p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Proof:



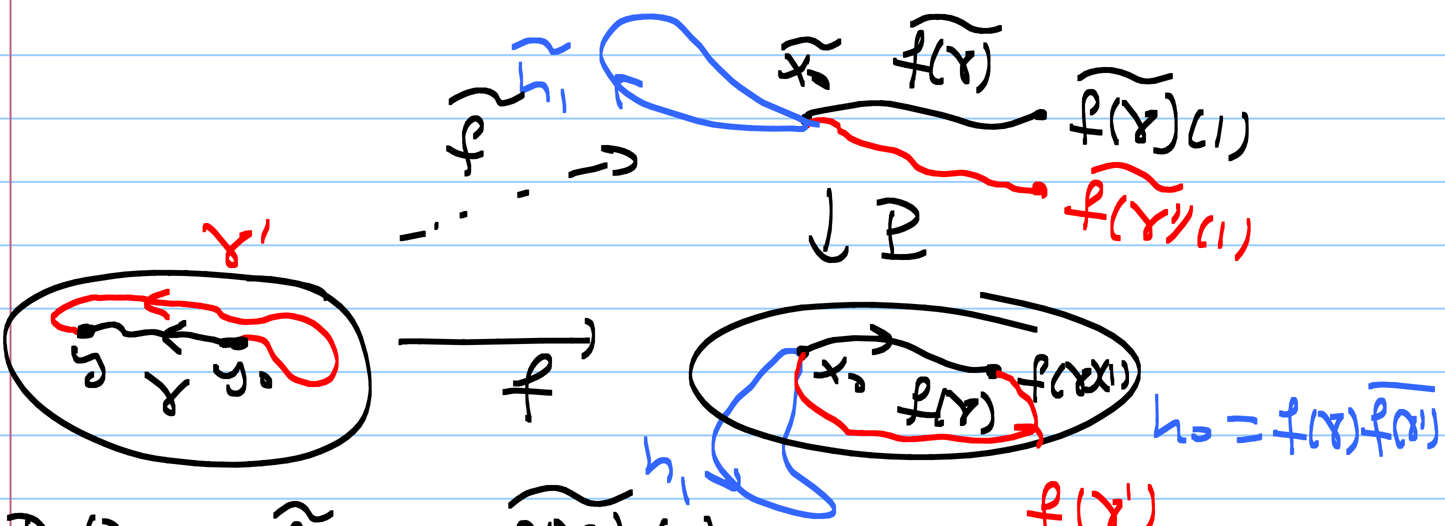
$p \circ \tilde f = f$

($\Rightarrow$) This direction is clear since if there is a lift $\tilde f$ then

$p_\# \circ \tilde f_\# = f_\#$ and thus

$$f_\#(\pi_1(Y,y_0)) = p_\#(\tilde f_\#(\pi_1(Y,y_0))) \subseteq p_\#(\pi_1(\tilde X,\tilde x_0))$$

($\Leftarrow$) Now assume that $f_\#(\pi_1(Y,y_0)) \subseteq p_\#(\pi_1(\tilde X,\tilde x_0))$.



Define $\tilde f(y) = \widetilde{f(\gamma)}(1)$.

<u>Well-definedness of $\tilde f$:</u> If γ' is another path from y_0 to y then we must show that $\widetilde{f(\gamma')}(1) = \widetilde{f(\gamma)}(1)$.

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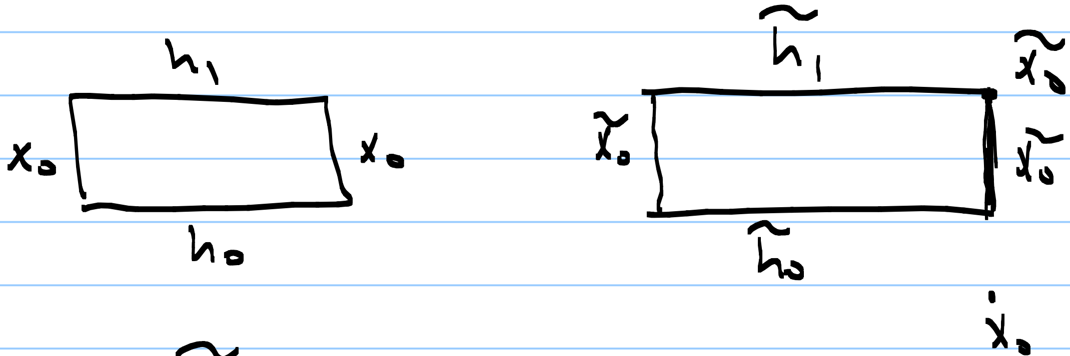
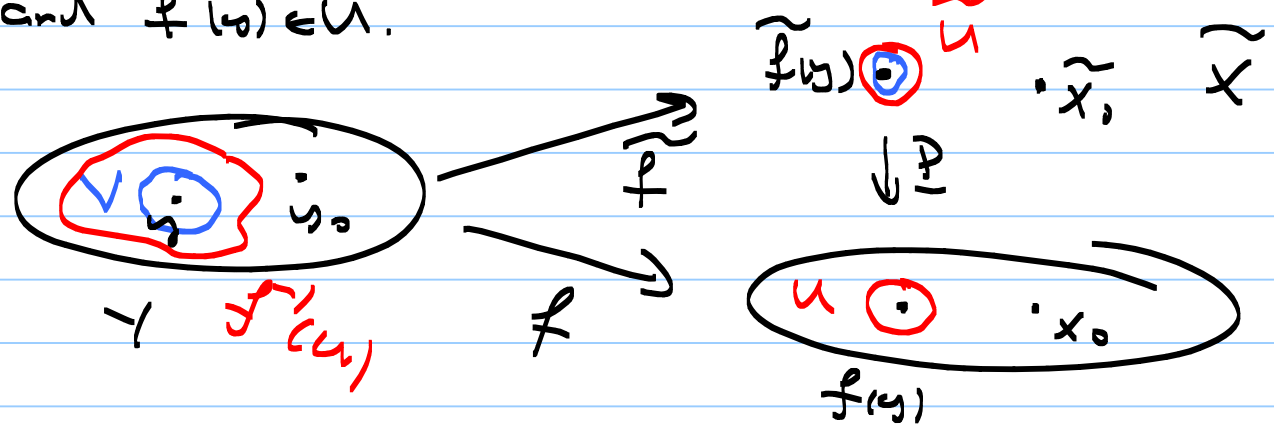
Now $f(\gamma')\cdot\overline{f(\gamma)}$ is a loop at x_0, representing a class
$$[h_0]\in f_{\#}(\pi_1(Y,y_0))\subseteq p_{\#}(\pi_1(\tilde{X},\tilde{x}_1)).$$

Hence there is a homotopy h_t of h_0 to a loop h_1 that has a lift $\tilde{h}_1$ in $\tilde{X}$ which is a loop based at $\tilde{x}_0$. By the homotopy lifting property h_t has a lift $\tilde{h}_t$. Since $\tilde{h}_1$ is a loop at $\tilde{x}_0$ so is $\tilde{h}_0$.



Hence, $\tilde{f}$ is well defined.

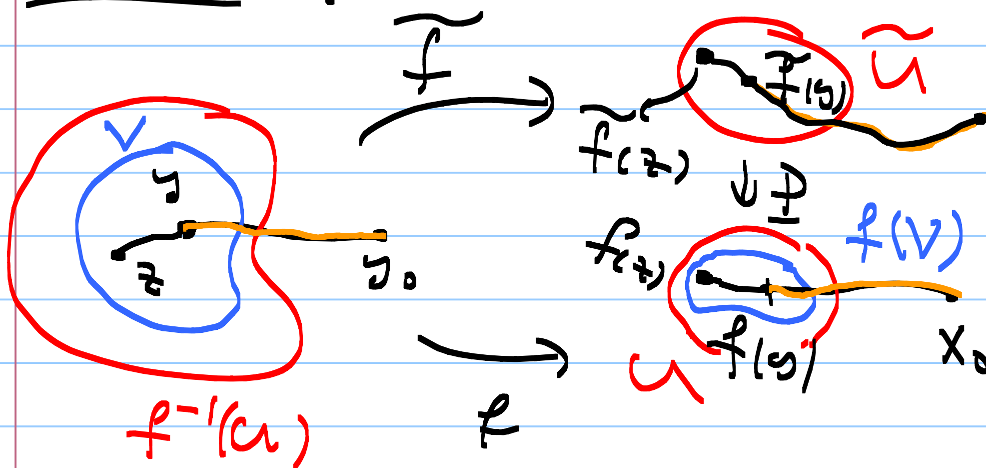
$\underline{\tilde{f}\text{ is continuous:}}$ Let $y\in Y$ and $U\subseteq X$ an open subset containing $f(y)$ such that there is some $\tilde{U}\subseteq\tilde{X}$ with $p:\tilde{U}\to U$ a homeomorphism and $\tilde{f}(y)\in\tilde{U}$.



$\underline{\text{must find:}}$ An open subset V in Y with $y\in V$ and $\tilde{f}(V)\subseteq\tilde{U}$.

Now since Y is locally path connected choose an open subset V so that $y \in V$, V is path connected and $f(V) \subseteq U$.

<u>Claim:</u> $\tilde{f}(V) \subseteq \tilde{U}$.

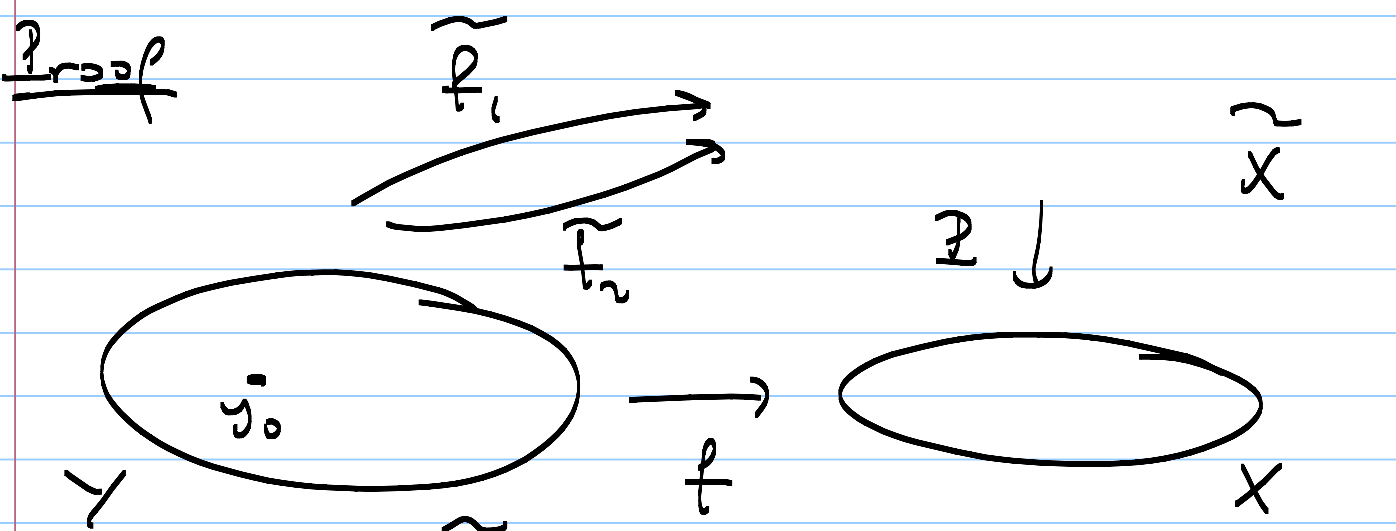


$\tilde{p} \circ \tilde{f} = f$

So, $\tilde{f}(V) \subseteq \tilde{U}$. Hence, $\tilde{f}$ is continuous. ∎

<u>Proposition:</u> Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$ with two lifts $\tilde{f}_1$ and $\tilde{f}_2$ from Y to $\tilde{X}$ that agree at one point y_0 of Y, then if Y is connected these two lifts agree on all of Y.

<u>Proof</u>



$p \circ \tilde{f}_i = f$, $i = 1, 2$.

$f_1(y_0) = f_2(y_0)$ is given.

<u>must show:</u> $f_1 = f_2$ or $f_1(y) = f_2(y)$ for all $y \in Y$.

Let A be the set of points in Y on which the two lifts agree:

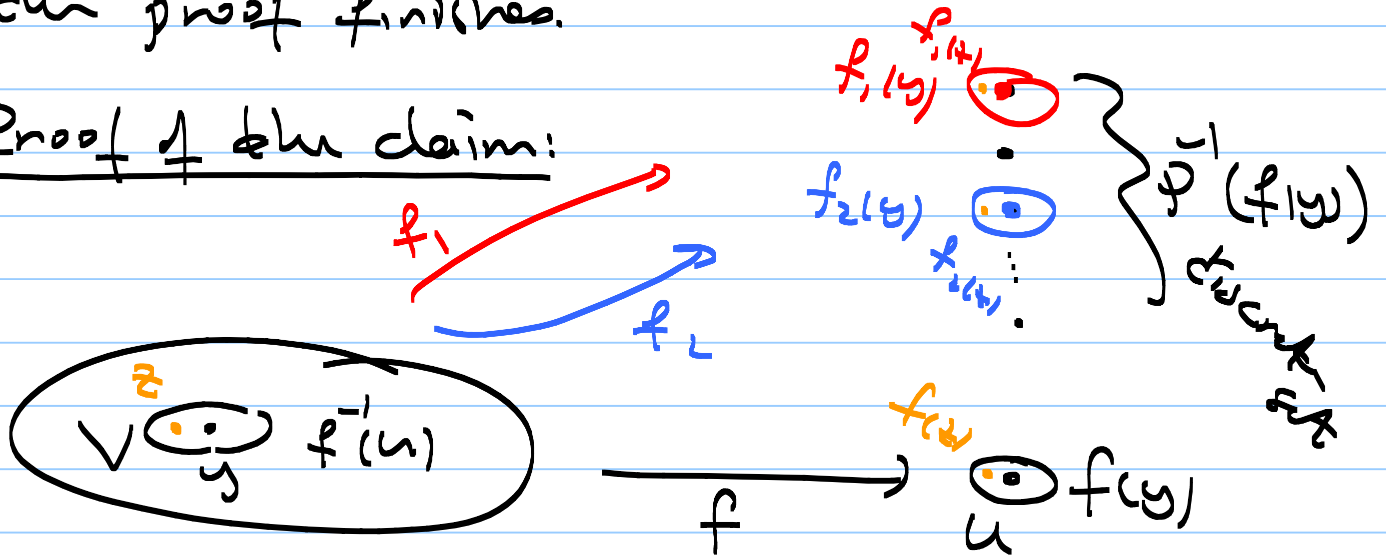
$$A = \{ y \in Y \mid f_1(y) = f_2(y) \}.$$

Clearly, $y_0 \in A$ and thus $A \neq \emptyset$.

<u>Claim:</u> A is both open and closed.

Note that the claim implies that $A = Y$ so that the proof finishes.

<u>Proof of the claim:</u>



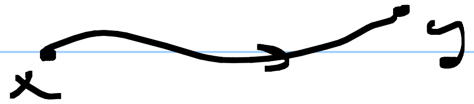
If $f_1(y) \neq f_2(y)$ then there is an open set $V = f^{-1}(u)$ so that $f_1(V) \cap f_2(V) = \emptyset$. In particular $f_1(z) \neq f_2(z)$ for all $z \in V$.

Hence, the set $Y \setminus A$ is open. Thus A is closed.
A similar argument shows that A is also open. ■

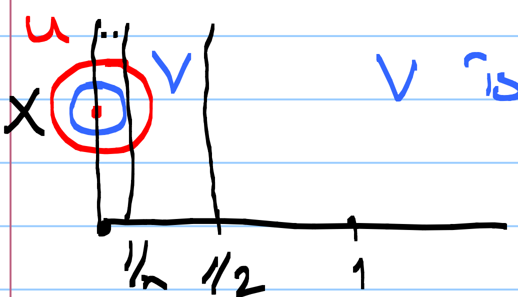
Video 29

Classification of Covering Spaces:

Let X be topological space, which is path connected, locally path connected and semilocally simply connected.

i) path connected: 

ii) locally path connected: $x \in X$, $x \in U$ open subset. Then there is another open subset V s.t. $x \in V \subseteq U$ and V is path connected.



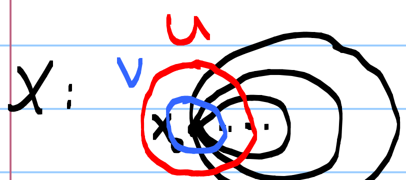
V is never path connected.

X is path connected but not locally path connected.

iii) Semi locally simply connected: For every $x \in X$ and open subset $x \in U$ there is some open subset $x \in V \subseteq U$ so that the homomorphism

$$\pi_1(V,x) \longrightarrow \pi_1(X,x)$$

is trivial.



$$X = \bigvee_{n=1}^{\infty} S^1_{1/n}$$

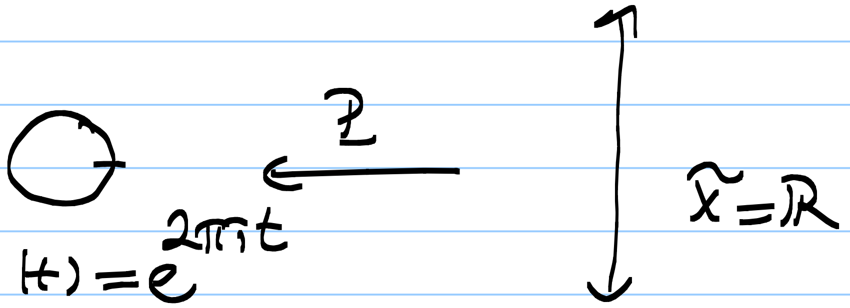
$$\underset{\substack{\cong \\ F_\infty}}{\pi_1(V,x_0)} \longrightarrow \pi_1(X,x_0) \text{ is injective}$$

Theorem: Let X be a path connected, locally path connected and semilocally simply connected topological space. Then X has a universal covering

$$p: \tilde{X} \longrightarrow X, \text{ i.e., a simply connected}$$

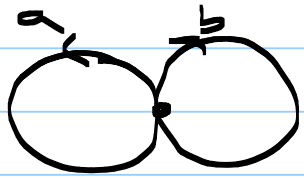
covering space.

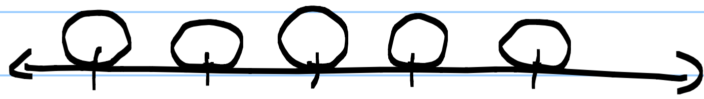
Remark: 1) $X = S^1$

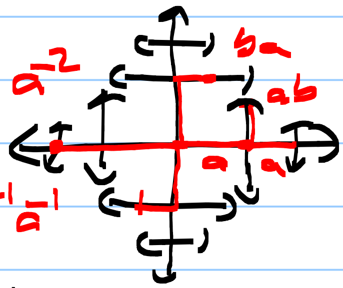


$$p(t) = e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$$

2) $X = S^1 \vee S^1$

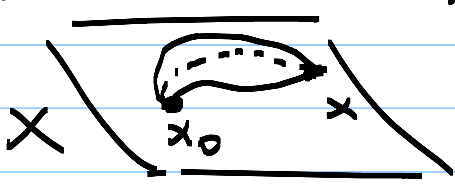






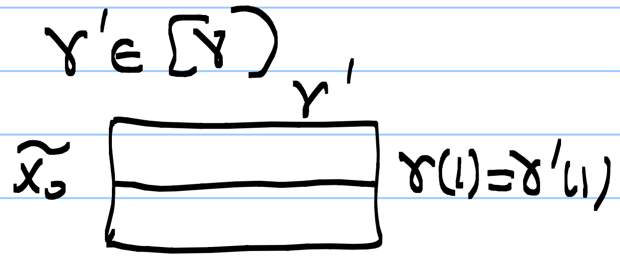
$\pi_1(X) = F_2 = \langle a, b \mid - \rangle \quad a, ab, b, abab, \ldots$

Observation: If X is a simply connected space then, there is a one to one correspondence between points of X as homotopy classes of paths starting at fixed point, where homotopies fixed the end points.

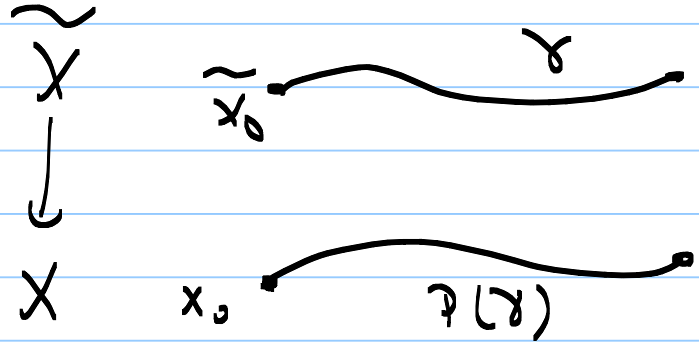


Idea: We may regard a simply connected space as the homotopy classes of paths in X starting at fixed point.

$$\tilde{X} = \{[\gamma] \mid \gamma(0) = \tilde{x}_0\}$$

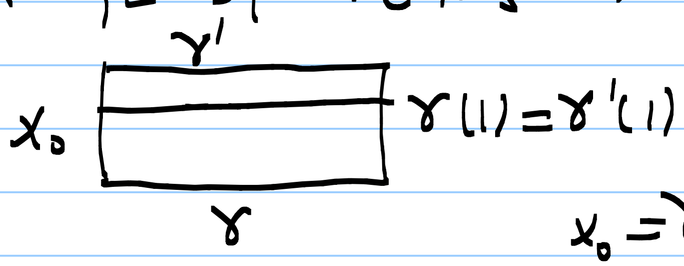


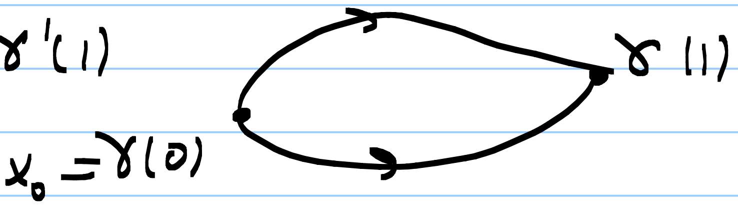
Any path in $\tilde{X}$ gives a path γ in X.



So let $\tilde{X}$ be the homotopy classes of path in X starting at a fixed point x_0, where homotopies fixes the end points of the paths at each level.

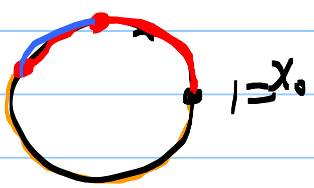
$$\tilde{X} = \{[\gamma] \mid \gamma : [0,1] \to X,\ \gamma(0) = x_0\}$$

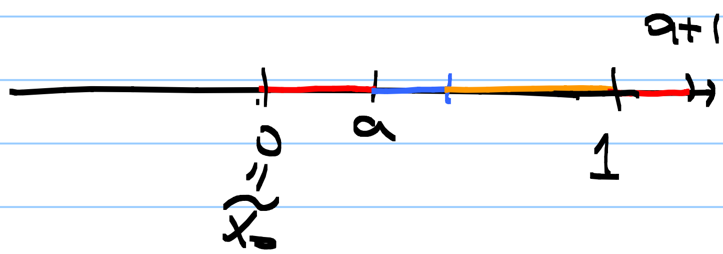




Example:

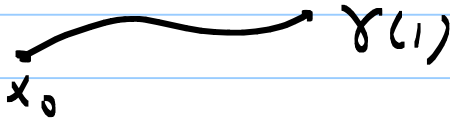
$(X, x_0) = (S^1, 1)$





How to define $P : \widetilde{X} \longrightarrow X$.

$[\gamma] \in \widetilde{X}$, $P([\gamma]) = \gamma(1)$



Clearly P is surjective, because X is path connected and thus for any $x \in X$ there is a path $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x_0$ and $\gamma(1) = x$.

Now we must put a topology on $\widetilde{X}$:

Let $\mathcal{U}$ denote the collection of path connected open subsets $U \subseteq X$ s.t.

$$\pi_1(U) \longrightarrow \pi_1(X) \text{ is trivial.}$$

Note that if $V \subseteq U$ another open subset then

$$\pi_1(V) \longrightarrow \pi_1(U) \longrightarrow \pi_1(X) \text{ is also trivial.}$$

Hence, $\mathcal{U}$ is a basis for the topology on X.

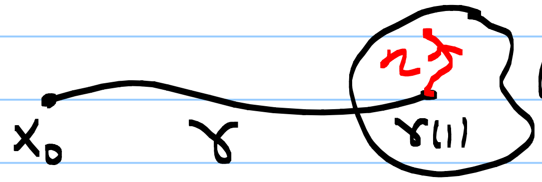
i) $x \in X$, $x \in W$ open. Then by assumption (s.l.s.c.) there is some $U \subseteq \mathcal{U}$ s.t.
$$x \in U \subseteq W.$$

ii) $x \in X$, $x \in U_1 \cap U_2$, $U_i \in \mathcal{U}$. Then $U_1 \cap U_2 \in \mathcal{U}$.

Next we'll describe a basis for a topology on $\widetilde{X}$:
Given $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

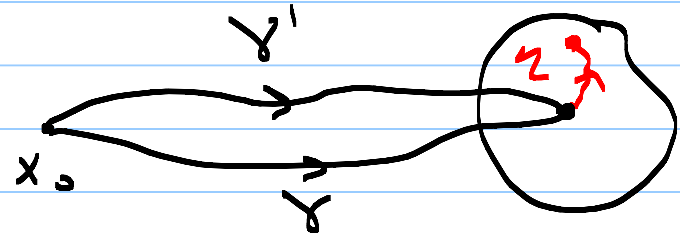
Video 3D

$U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \}$



<u>Observations</u> 1) $U_{[\gamma]}$ depends only on the homotopy class of γ. In other words, if $\gamma' \in [\gamma]$ then

$U_{[\gamma']} = U_{[\gamma]}$.



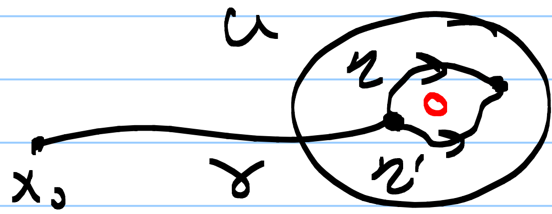
$[\gamma \cdot \eta] = [\gamma' \cdot \eta]$

2) Since U is path connected the map

$P : U_{[\gamma]} \longrightarrow U, \quad [\gamma \cdot \eta] \longmapsto \eta(1)$, is onto.

3) Moreover, P is injective.

Assume $[\gamma \cdot \eta]$ and $[\gamma \cdot \eta']$ are in $U_{[\gamma]}$ so that $P([\gamma \cdot \eta]) = \eta(1) = \eta'(1) = P([\gamma \cdot \eta'])$.



Since $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial the paths $[\eta]$ and $[\eta']$ are homotopic in X and thus $\gamma \cdot \eta$ and $\gamma \cdot \eta'$ are homotopic via a homotopy fixing the end points.

Conclusion: $P: U_{[\gamma]} \to U$ is a bijection.

Any point $[\gamma] \in \tilde{X}$ satisfies $[\gamma] \in U_{[\gamma]}$.

Now using these we may put a topology on $\tilde{X}$.

Claim: The collection

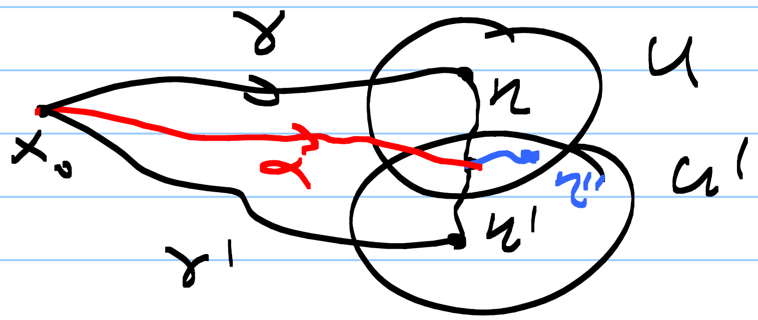
$\tilde{\mathcal{U}} = \{ U_{[\gamma]} \mid U \in \mathcal{U}, [\gamma] \in \tilde{X} \}$ is a basis

for a topology on $\tilde{X}$.

In this case each bijection $P: U_{[\gamma]} \to U$ becomes a homeomorphism.

We already showed that any $[\gamma] \in \tilde{X}$ lies in $U_{[\gamma]}$. Now let

$$[\alpha] \in U_{[\gamma]} \cap U'_{[\gamma']}.$$



$[\alpha] = [\gamma \cdot \eta] = [\gamma' \cdot \eta']$

Then $[\alpha] \in U \cap U'_{[\alpha]} \underset{?}{\subseteq} U_{[\gamma]} \cap U_{[\gamma']}$ because

$\underline{[\alpha \cdot \eta'']} = \underline{[\gamma \cdot \eta \cdot \eta'']} = \underline{[\gamma' \cdot \eta' \cdot \eta'']}$

In particular, each restriction $P: U_{[\gamma]} \to U$

is a homeomorphism.

We must show: 1) P is continuous

2) P is covering map

3) $\tilde{X}$ is simply connected.

P is continuous: Take any $x \in X$ and $x \in V \subseteq X$ an open subset. Then by the s.l.s.c. assumption there is an open subset U s.t. $x \in V \subseteq U \subseteq X$ with $U \in \tilde{\mathcal{U}}$. Now $\exists\, [\gamma] \in \tilde{X}$ s.t.

$P([\gamma]) = x$, i.e., $\gamma : [0,1] \to X$, $\gamma(0) = x_0$, $\gamma(1) = x$

$P : U_{[\gamma]} \to U$ is a bijection and thus

$P(U_{[\gamma]}) = U \subseteq V$. Hence, P is continuous.

2) **P is a covering map:** Take any $x \in X$. Choose some $U \subseteq \mathcal{U}$ with $x \in U$. Then for any $[\gamma] \in \tilde{X}$ with $P([\gamma]) = \gamma(1) = x$, the open

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set $U_{[\gamma]}$ maps homeomorphically onto U.

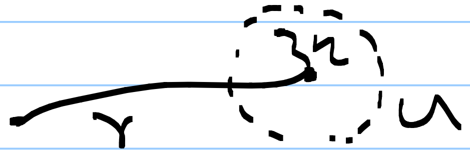
We must show:

a) $p^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]}$

b) If $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$ then $U_{[\gamma]} = U_{[\gamma']}$.

Proof of a) Let $[\gamma] \in p^{-1}(U)$. Then $p([\gamma]) = \gamma(1) \in U$.

In particular, $[\gamma] \in U_{[\gamma]}$ and $p(U_{[\gamma]}) = U$.

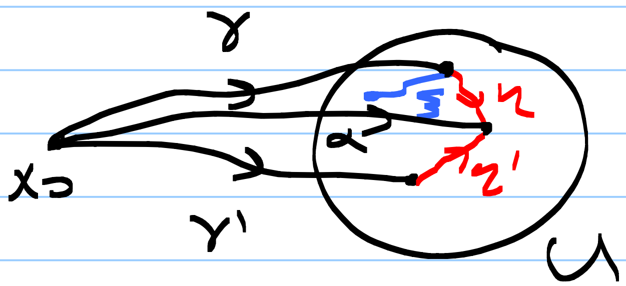


Hence, $p^{-1}(U) = \bigcup_{\gamma(1)\in U} U_{[\gamma]}$.

Proof of b) Assume that $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$ and

$\alpha \in U_{[\gamma]} \cap U_{[\gamma']}$. Then there are paths η and η' in U so that

$\gamma(1) = \eta(0)$, $\gamma'(1) = \eta'(0)$ and $[\gamma\cdot\eta] = [\alpha] = [\gamma'\cdot\eta']$.



So $[\gamma\cdot\eta\cdot\bar{\eta}'\cdot\bar{\gamma}'] = e$ in $\pi_1(X, x_0)$.

Hence, $[\gamma] = [\gamma'\cdot\eta'\cdot\bar{\eta}]$. Thus, $[\gamma\cdot\xi] = [\gamma'\cdot\eta'\cdot\bar{\eta}\cdot\xi]$ for any path ξ in U with $\gamma(1) = \xi(0)$.

However, $[\gamma\cdot\xi]$ is a typical element of $U_{[\gamma]}$ and $[\gamma'\cdot(\eta'\cdot\bar{\eta}\cdot\xi)]$ is in $U_{[\gamma']}$. Thus, $U_{[\gamma]} \subseteq U_{[\gamma']}$.

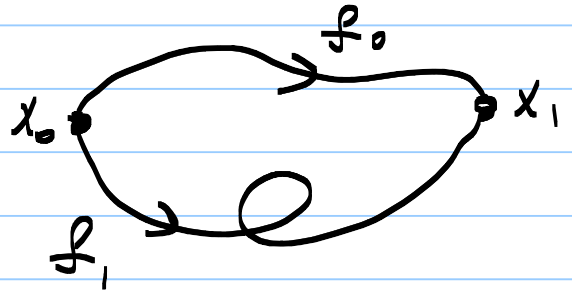
Similarly, $U_{[\gamma']} \subseteq U_{[\gamma]}$ and thus $U_{[\gamma]} = U_{[\gamma']}$.

In the above proof we used the following fact:

<u>Fact</u>: For any two path f_0 and f_1 in any space from x_0 to x_1, we have

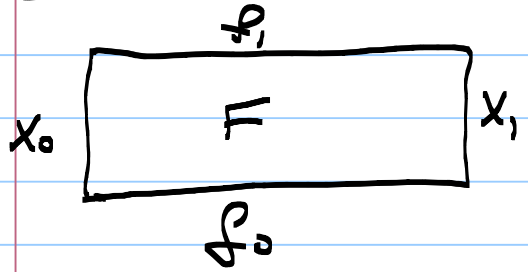
$$f_0 \sim f_1 \ (\text{rel}\{0,1\}) \text{ if and only if } [f_0 \cdot \bar{f}_1] = e \text{ in}$$

$\pi_1(X, x_0)$.



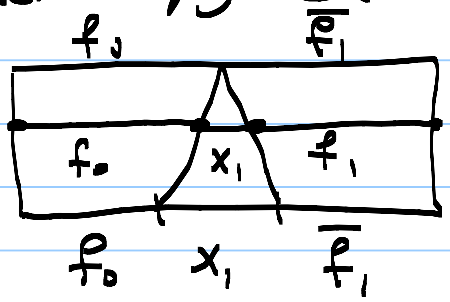
<u>Proof of the fact</u>:

($\Longrightarrow$) Assume that $f_0 \sim f_1$, rel$\{0,1\}$.

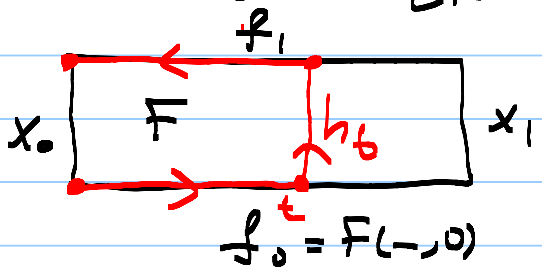


<u>must show</u>: $[f_0 \cdot \bar{f}_1] = e$ in $\pi_1(X, x_0)$.

First note that $[f_0 \cdot x_1^* \cdot \bar{f}_1] = [f_0 \cdot \bar{f}_1]$ in $\pi_1(X, x_0)$, where x_1^* is the constant path at x_1. To see this just consider the homotopy described by the diagram below:



Now we show $[f_0 \cdot x_1^* \cdot \bar{f}_1] = e$ in $\pi_1(X, x_0)$.



$h_t : I \longrightarrow X, \quad t \in [0,1]$

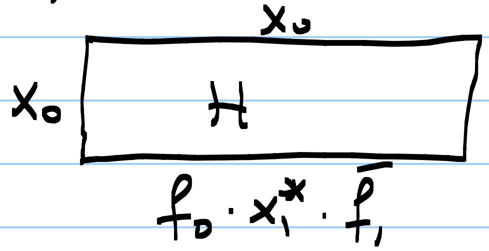
$h_1 = f_0 \cdot x_1^* \cdot f_1, \quad h_0 = x_0^*$

$$h_t(s) = \begin{cases} F(4s, 0), & 0 \leq s \leq t/4 \\ F\left(t, \frac{4-t-4s}{4-2t}\right), & t/4 \leq s \leq 1 - t/4 \\ F(4-4s, 1), & 1 - t/4 \leq s \leq 1. \end{cases}$$

$$h_1(s) = \begin{cases} F(4s, 0), & 0 \leq s \leq 1/4 \\ F\left(1, \frac{3-4s}{2}\right), & 1/4 \leq s \leq 3/4 \\ F(4-4s, 1), & 3/4 \leq s \leq 1 \end{cases}$$

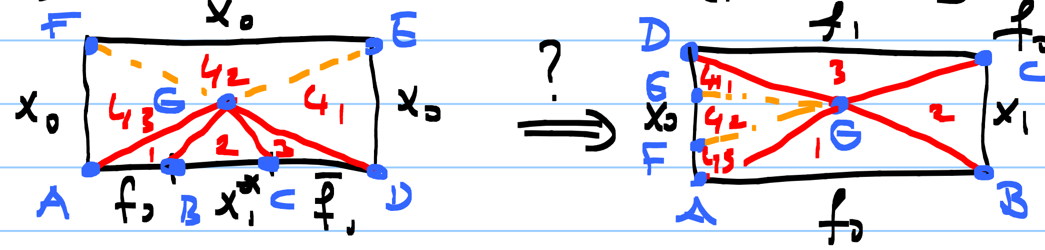
$$= \begin{cases} f_0(4s), & 0 \leq s \leq 1/4 \\ x_1, & 1/4 \leq s \leq 3/4 \\ f_1(4-4s), & 3/4 \leq s \leq 1. \end{cases} \sim f_0 \cdot x_1^* \cdot \bar{f}_1$$

Also, $h_0(s) = x_0$, for all s.



, so that $[f_0 \cdot x_1^* \cdot \bar{f}_1] = e$ in $\pi_1(X, x_0)$.

($\Leftarrow$) Now assume that $[f_0 \cdot x_1^* \cdot \bar{f}_1] = e$. Must show $f_0 \simeq f_1 \text{ rel}\{0,1\}$



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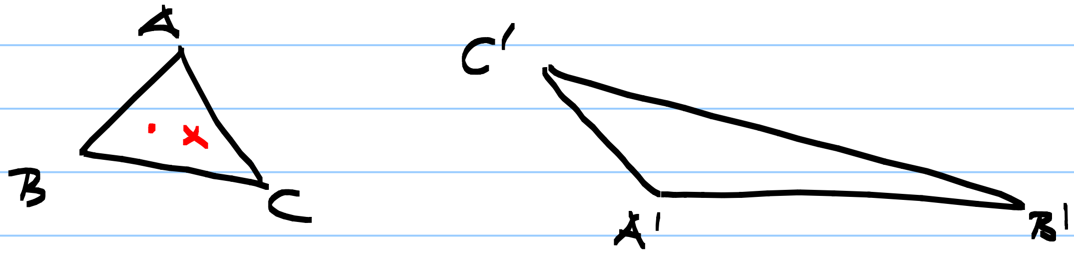
Now we may write a homeomorphism φ which is linear on each triangle mapping every vertex in the first rectangle to the vertex in the second rectangle with the same name.

$F : I\times I \to X \Rightarrow F\circ\varphi^{-1} : I\times I \to X$



$x = t_1 A + t_2 B + t_3 C \longmapsto \varphi(x) = t_1 A' + t_2 B' + t_3 C'$

$t_1, t_2, t_3 \geq 0, \ \sum t_i = 0$

(t_1, t_2, t_3) is called the barycentric coordinates of x.

This finishes the proof of that $P : \tilde{X} \to X$ is a covering map.

Now we need to prove that $\tilde{X}$ is simply connected.

$\tilde{X} = \{[\gamma] \mid \gamma : [0,1] \to X, \ \gamma(0) = x_0\}$

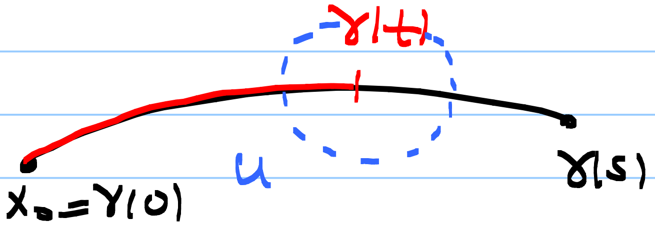
$x_0^* : [0,1] \to X$ the constant path at x_0.

$[x_0^*] \in \tilde{X}$ can be considered as a base point.

<u>$\tilde{X}$ is connected</u>: Take any point $[\gamma] \in \tilde{X}$.

So, $\gamma : [0,1] \to X$ is so that $\gamma(0) = x_0$.
For any $t \in [0,1]$ define the path

$\gamma_t : [0,1] \to X, \quad \gamma_t(s) = \begin{cases} \gamma(s), & 0 \le s \le t \\ \gamma(t), & t \le s \le 1. \end{cases}$



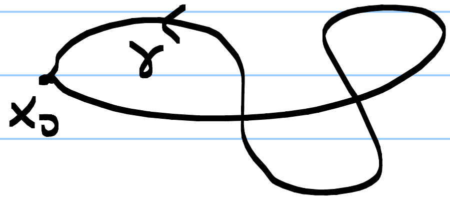
Then we get a path in $\tilde{X}$: $t \mapsto [\gamma_t]$

γ_0 = constant path at x_0
$\gamma_1 = \gamma$

So the path $t \mapsto [\gamma_t]$ in $\tilde{X}$ joins $[x_0^*]$ to $[\gamma]$. This path is continuous by the local description of topology on $\tilde{X}$ and thus $\tilde{X}$ is path connected.

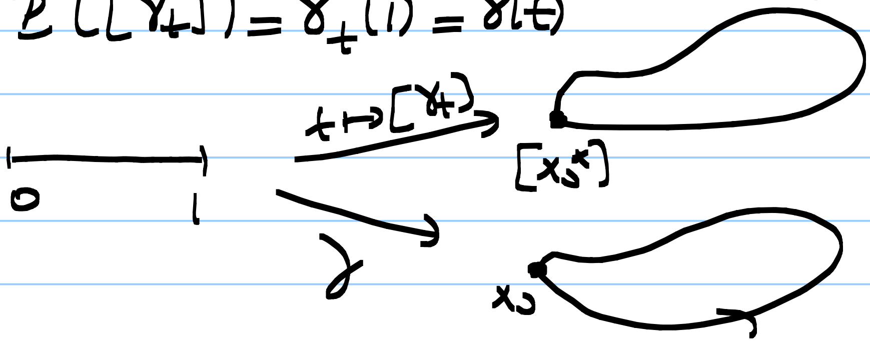
$\underline{\tilde{X} \text{ is simply connected}}$: Since the homomorphism

$p_\# : \pi_1(\tilde{X}, [x_0^*]) \to \pi_1(X, x_0)$ is injective, it is enough to show that $\operatorname{Im} p_\#$ is trivial. Let $[\gamma]$ be in the image of $p_\#$. So γ is a loop at x_0.



Then the path $t \mapsto [\gamma_t]$ is a lift of γ to the covering $\tilde{X} \to X$, because

$p([\gamma_t]) = \gamma_t(1) = \gamma(t)$



$$\begin{array}{c} \tilde{X} \\ \downarrow p \\ X \end{array}$$

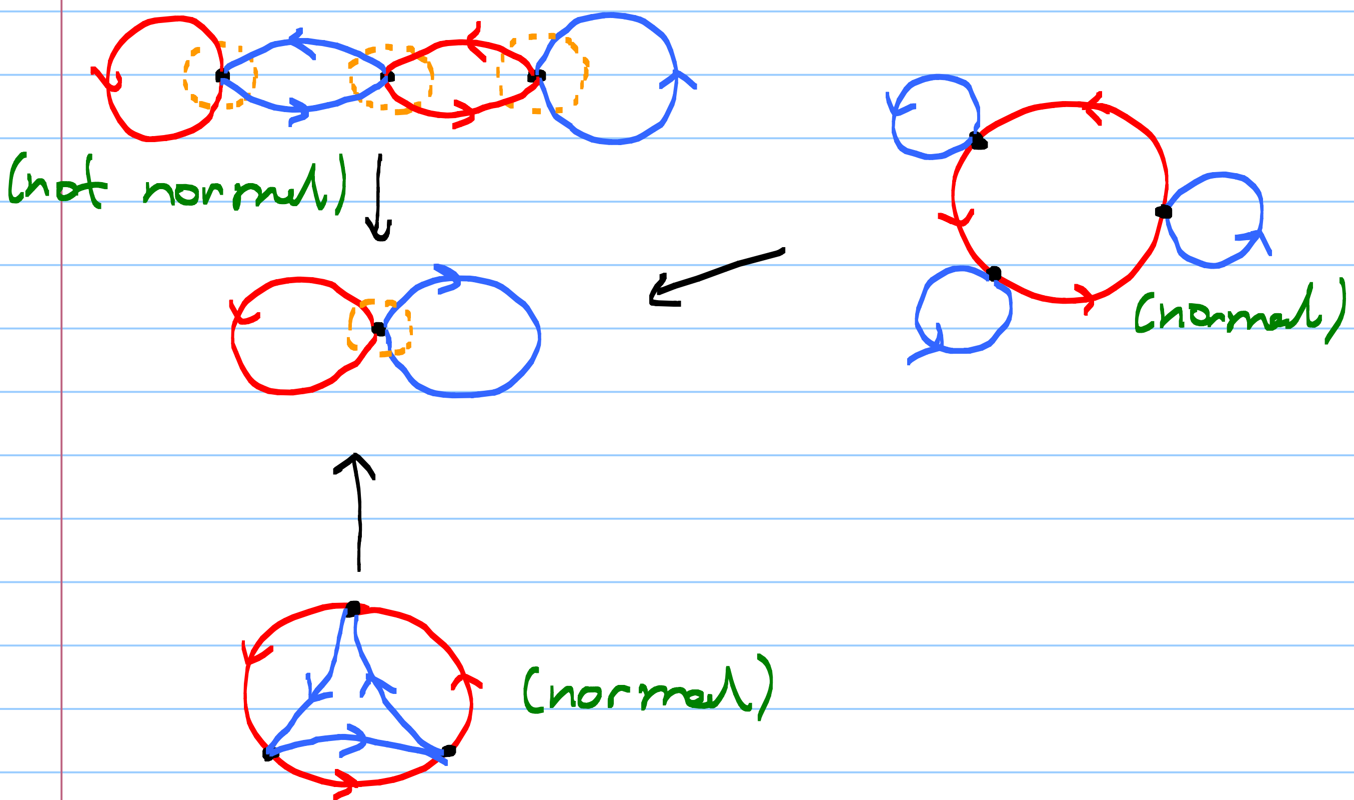
Since $t \longmapsto [\gamma_t]$ is a loop at $[x_0^*]$ the end points of this path must be the same.

$t=0 \Rightarrow x_0^*$, $t=1 \Rightarrow [\gamma]$

$\Rightarrow [x_0^*] = [\gamma] \Rightarrow [\gamma] = e \in \pi_1(X, x_0)$.

So we've proved that any topological space (path connected, l.c.p., s.l.s.c) has a universal covering space.

Remark: In practice we construct covering spaces as follows: Note that for a simply connected space any connected covering of it is itself because its fundamental group is trivial and thus any covering is 1-fold.



Proposition: Suppose X is path connected, locally path connected, and semilocally simply connected. Then for

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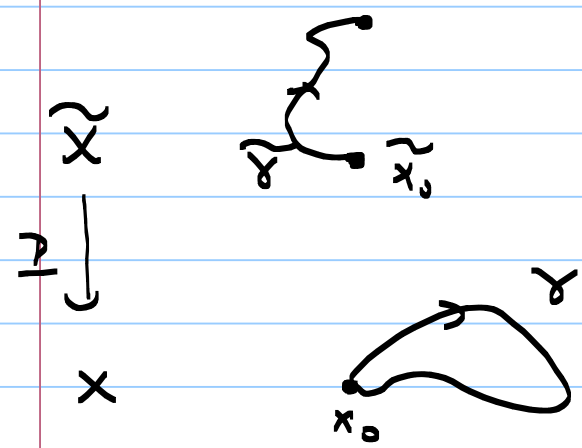
every subgroup $H \leq \pi_1(X, x_0)$ there is a covering space $P: X_H \to X$ such that

$$P_\#(\pi_1(X_H, \tilde{x}_0)) = H, \text{ for a suitable chosen}$$

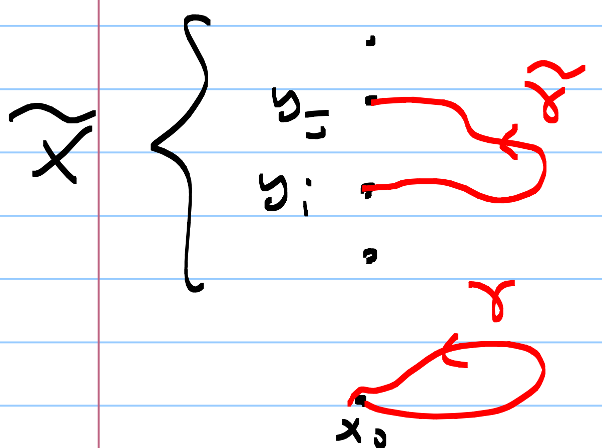
base point $\tilde{x}_0 \in X_H$.

Proof: The main observation is the following. A loop γ lifts to a loop in the covering space if and only if $[\gamma]$ belongs to $H = P_\#(\pi_1(X, \tilde{x}_0))$.

In particular, if $\tilde{X}$ is simply connected then no lift of a loop γ with $[\gamma] \neq e$ is a loop in $\tilde{X}$.



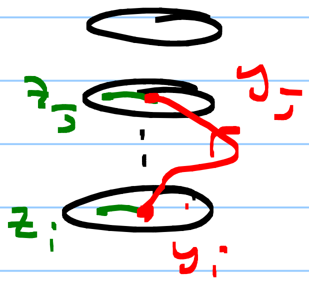
X_H will be constructed as a quotient of $\tilde{X}$:

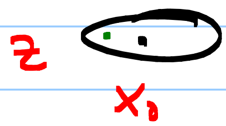


$\tilde{X} \to X$ universal cover

If $y_i, y_j \in P^{-1}(x_0)$ then we say $y_i \sim y_j$ if and only if the image of the homotopy class $\tilde{\gamma}$ in $\pi_1(X, x_0)$ is in H.

$y_i \sim y_j \iff [\gamma] \in H \leq \pi_1(X, x_0)$.





$U \subseteq X$ basis neighborhood for the universal cover.

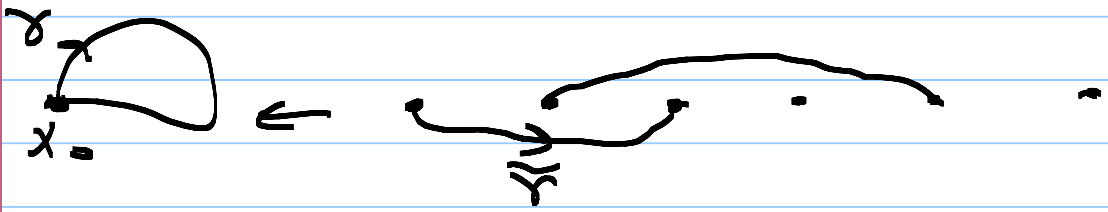
So let X_H be the quotient space $\tilde{X}/\sim$, where $\sim$ is as defined above. Note that for any basis neighborhood U in X two components $\tilde{U}_i$ and $\tilde{U}_j$ of $p^{-1}(U)$ are either identified by a homeomorphism or no points of $\tilde{U}_i$ and $\tilde{U}_j$ are identified.

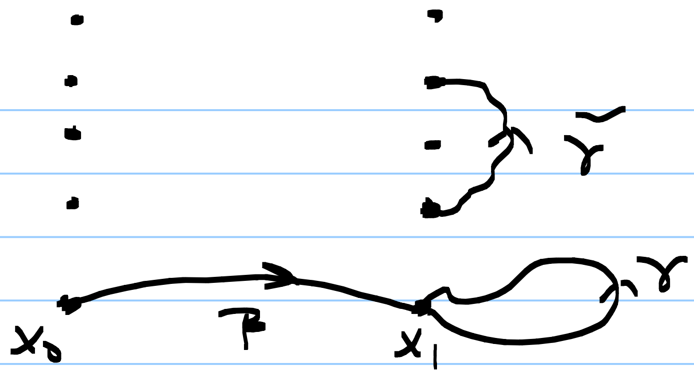
$$\tilde{U}_i \xrightarrow{p_{\tilde{U}_i}} U, \quad \tilde{U}_j \xrightarrow{p_{\tilde{U}_j}} U$$

$$\tilde{U}_i \xrightarrow{p^{-1}_{\tilde{U}_j} \circ p_{\tilde{U}_i}} \tilde{U}_j$$

Note that $X_H \longrightarrow X$ is still a covering space.

$p_\#(\pi_1(X_H, y_0)) = H$ because we identified the end points of any lift of $[\gamma] \in \pi_1(X, x_0)$ if and only if $[\gamma] \in H$.





$[\beta \cdot \gamma \cdot \bar{\beta}] \in H$?

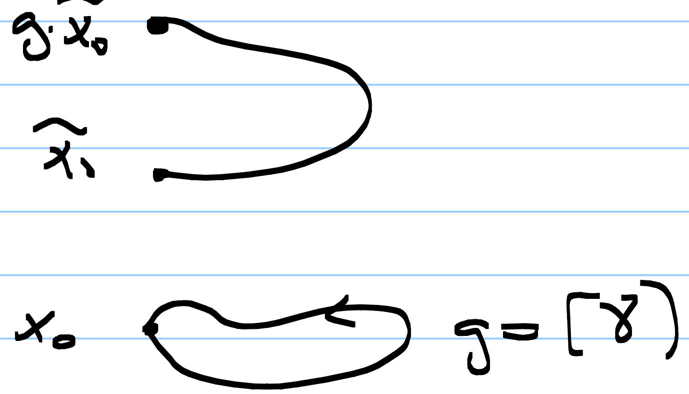
This finishes the proof. ∎

<u>Remark:</u> $G = \pi_1(X, x_0)$, $\tilde{X} \longrightarrow X$ universal cover.

Then G acts on $\tilde{X}$ and $\tilde{X}/G = X$.

Moreover, if $H \leq G$, then $X_H = \tilde{X}/H$.

$G = \pi_1(X, x_0)$



<u>Definition:</u> Let $p_1 : \tilde{X}_1 \longrightarrow X$ and $p_2 : \tilde{X}_2 \longrightarrow X$ be two covering spaces. We'll say that they are isomorphic if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$.

$(\iff f^{-1} \circ p_1 = p_2)$

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Proposition: If X is path connected and locally path connected, then two path connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ taking a base point $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p_2^{-1}(x_0)$ if and only if

$$p_{1\#}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2\#}(\pi_1(\tilde{X}_2, \tilde{x}_2)).$$

$$\begin{tikzcd} \tilde{X}_1 \quad \tilde{x}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{x}_2 \quad \tilde{X}_2 \arrow[dl, "p_2"] \\ & X \quad x_0 & \end{tikzcd}$$

Proof: ($\Longrightarrow$) Suppose there is an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ so that $p_1 = p_2 \circ f$.

Hence, $p_{1\#}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_{2\#} \circ f_\#)(\pi_1(\tilde{X}_1, \tilde{x}_1))$

$$= p_{2\#}(\pi_1(\tilde{X}_2, \tilde{x}_2))$$

($\Longleftarrow$) Conversely if $p_1(\pi_1(\tilde{X}, \tilde{x}_1)) = p_2(\tilde{X}_2, \tilde{x}_2)$ then by the lifting criterion the map

$$\begin{tikzcd} & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[r, "p_1"'] \arrow[ur, dotted, "\exists! f"] & (X, x_0) \end{tikzcd}$$

there is a unique $f : (\tilde{X}_1, \tilde{x}_1) \to (X_2, \tilde{x}_2)$ so that $p_2 \circ f = p_1$.

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Similarly, there is unique $g : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$ such that $p_1 \circ g = p_2$.

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[r, "f"] \arrow[d, "p_1"'] \arrow[rr, bend right, "\mathrm{id}_{\tilde{X}_1}"'] & (\tilde{X}_2, \tilde{x}_2) \arrow[r, "g"] \arrow[dl, "p_2"] \arrow[dr, "p_2"'] & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\
(X, x_0) \arrow[rr, equal, "\mathrm{id}"'] & & (X, x_0)
\end{tikzcd}$$

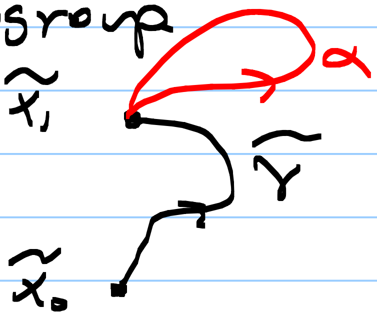
$y \mapsto f(y) \mapsto g(f(y))$, $x \mapsto x$

$\Rightarrow f \circ g = \mathrm{id}_{\tilde{X}_1}$. Similarly, $g \circ f = \mathrm{id}_{\tilde{X}_2}$. Hence, f and g are homeomorphism. ∎

Now we are ready to state the classification theorem.

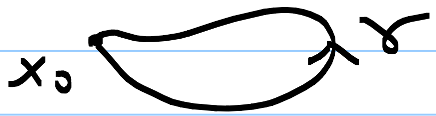
<u>Theorem</u>: Let X be path connected, locally path connected and semilocally simply connected. Then there is a bijection between the set of base point-preserving isomorphism classes of path-connected covering spaces
$$p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$$
and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$. If the base points are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

Proof: Note that we just need to prove the last statement. For a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ changing the basepoint within $p^{-1}(x_0)$ corresponds exactly to changing the $p_\#(\pi_1(\tilde{X},\tilde{x}_0))$ to a conjugate subgroup



$$H = p_\#(\pi_1(X,\tilde{x}_0))$$

$$[\alpha]\in\pi_1(\tilde{X}_1,\tilde{x}_1)$$



$$H_1 = p_\#(\pi_1(X,\tilde{x}_1))$$

$$\pi_1(X,\tilde{x}_0) = \{[\tilde{\gamma}.\alpha.\bar{\tilde{\gamma}}] \mid [\alpha]\in\pi_1(\tilde{X},\tilde{x}_1)\}.$$

$$p_\# \downarrow$$

$$H \qquad \{[\gamma]\cdot p_\#([\alpha])\,[\gamma]^{-1} \mid [\alpha]\in\pi_1(\tilde{X},\tilde{x}_1)\}$$

$$= [\gamma]H_1[\gamma]^{-1}$$

$$\Rightarrow H = [\gamma]H_1[\gamma]^{-1}.$$

The converse is left as an exercise! ∎

Remark: Let $H=(e)\le\pi_1(X,x_0)$. Any conjugate of H is itself. Thus the simply connected covering space we constructed earlier is unique upto isomorphism. Hence, we may call the universal covering space.

Deck Transformations and Group Actions:

For a covering space $p:\tilde{X}\to X$ the isomorphisms $\tilde{X}\to\tilde{X}$ are called deck transformations or covering transformations.

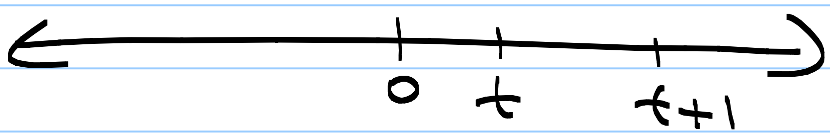
$$\begin{tikzcd} \tilde{x} \arrow[rr, bend left] \arrow[dr, bend right] & \tilde{X} \arrow[r, "f"] \arrow[d, "p"'] & \tilde{X} \arrow[dl, "p"] & f(\tilde{x}) \arrow[dll, bend left] \\ & X & & \\ & x & & \end{tikzcd}$$

$$p\circ f = p$$

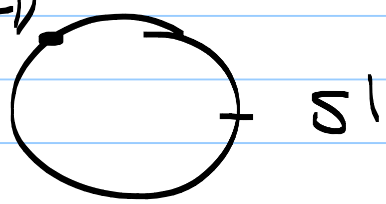
$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "id"'] & X \end{tikzcd}$$

Hence, deck transformations are just the lifts of the $id: X\to X$ to $\tilde{X}$.

Example: $\mathbb{R}\to S^1$, $p(t)=(\cos 2\pi t, \sin 2\pi t)$



$p(t)=p(t+1)$



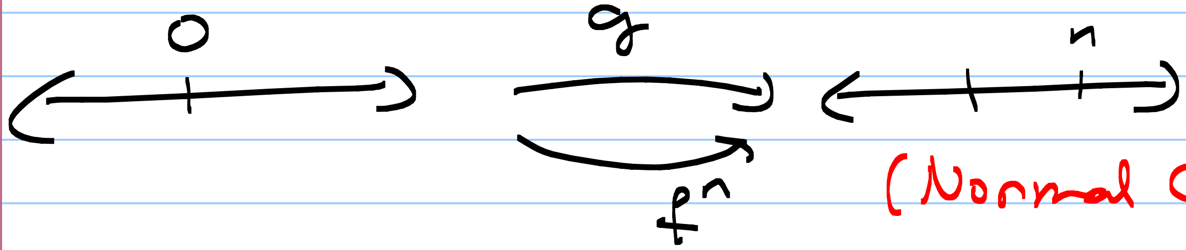
Hence, $f(t)=t+1$, $t\in\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$ is a covering transformation.

$f^n:\mathbb{R}\to\mathbb{R}$, $f^n(t)=f(f(\cdots(f))\cdots)(t)=t+n$

is a covering transformation for all n.

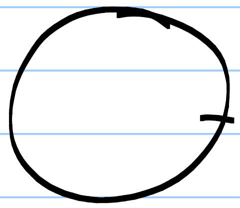
Let $G(\tilde{X})$ denote the group of deck transformations.

Then for the above covering $G(\mathbb{R}) \simeq \mathbb{Z}$.



(Normal Cover!)

$p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$



$p(0) = 1$

$g = f^n \implies G(\mathbb{R}) \simeq \mathbb{Z}$.

$G(\mathbb{R} \xrightarrow{p} S^1) \simeq \mathbb{Z}$

<u>Definition</u>: A covering space $p: \tilde{X} \to X$ is called <u>normal</u> if for each $x \in X$ and each pair of points $\tilde{x}, \tilde{x}'$ over x there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

$\cdot\ \tilde{x}'$

$\cdot\ \tilde{x}$ $\qquad f(\tilde{x}) = \tilde{x}'$

$\downarrow p$

$\cdot\ x$

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Proposition: Let $\tilde{p}: (\tilde{X}, \tilde{x}_1) \to (X, x_1)$ be a path connected covering space of path connected, locally path connected space X, and let H be the subgroup $p_\#(\pi_1(\tilde{X}, \tilde{x}_1)) \leq \pi_1(X, x_1)$. Then:

a) This covering space is normal if and only if H is a normal subgroup of $\pi_1(X, x_1)$.

b) $G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_1)$.

In particular, $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$ if $\tilde{X}_0$ is a normal covering. Hence for the universal cover $\tilde{X} \to X$ we have $G(\tilde{X}) \cong \pi_1(X, x_1)$.

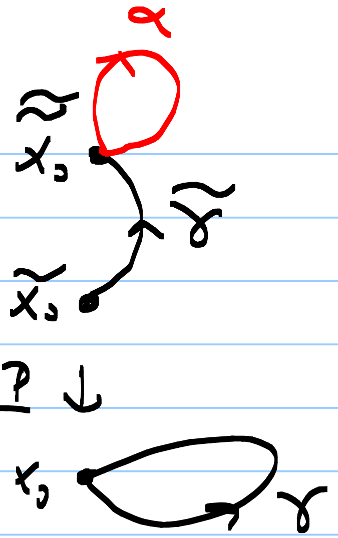
Corollary Assume the above set up. Then the cover $p: \tilde{X} \to X$ is normal if and only if for any $[\gamma] \in \pi_1(X, x_0)$ we have either all lifts of γ are loops or all lifts of γ are non-loops.

Proof of the Proposition:

a) First assume that the covering is normal. So we must prove that $H = p_\#(\pi_1(\tilde{X}, \tilde{x}_1))$ is a normal subgroup of $\pi_1(X, x_0)$. It is enough to prove that

$$[\gamma] H [\gamma]^{-1} = H \quad \text{for any } [\gamma] \in \pi_1(X, x_0).$$

Let $\tilde{\gamma}$ be the unique lift of γ starting at $\tilde{x}_0$.



$$\tilde{\gamma} \cdot \pi_1(\tilde{X}, \tilde{\tilde{x}}_0)\, \overline{\tilde{\gamma}} \simeq \pi_1(\tilde{X}, \tilde{x}_0).$$
$$\big\downarrow P_\#$$
$$[\gamma] \cdot H_1 \cdot [\gamma]^{-1} = H, \text{ where}$$
$$H_1 = P_\#(\pi_1(\tilde{X}, \tilde{\tilde{x}}_0))$$

By assumption there is a deck transformation $f \in G(\tilde{X})$ so that $f(\tilde{x}_0) = \tilde{\tilde{x}}_0$.

In particular, $f_\#(\pi_1(\tilde{X}, \tilde{x}_0)) = \pi_1(\tilde{X}, \tilde{\tilde{x}}_0)$, since f is a homeomorphism. Since $P = P \circ f$ we

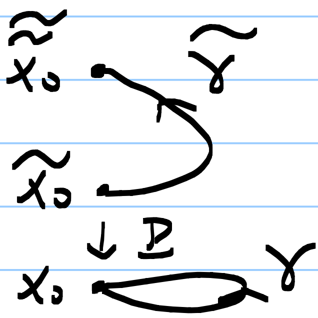
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "P"'] & & \tilde{X} \arrow[dl, "P"] \\ & X & \end{tikzcd}$$

get

$$H = P_\#(\pi_1(\tilde{X}, \tilde{x}_0)) = P_\#(f_\#(\pi_1(\tilde{X}, \tilde{x}_0)))$$
$$= P_\#(\pi_1(\tilde{X}, \tilde{\tilde{x}}_0))$$
$$= H_1.$$

Hence, $H = H_1 = [\gamma] H [\gamma]^{-1}$, where $[\gamma] \in \pi_1(X, x_0)$ is any element. Thus $H \lhd \pi_1(X, x_0)$.

Now assume that H is normal in $\pi_1(X, x_0)$. Let $\tilde{x}_0$ and $\tilde{\tilde{x}}_0$ lie above x_0 and choose a path $\tilde{\gamma}$ joining $\tilde{x}_0$ to $\tilde{\tilde{x}}_0$. Let $\gamma = P(\tilde{\gamma})$.



Let $H_1 = P_\#(\pi_1(\tilde{X}, \tilde{\tilde{x}}_0))$ as above.

Then $H_1 = [\gamma]^{-1} H [\gamma] = H$

Consider the diagram below:

$$\begin{tikzcd}
 & (\tilde X, \tilde{\tilde{x}}_0) \arrow[d, "p"] \\
(\tilde X, \tilde x_0) \arrow[ur, dashed, "\exists! f"] \arrow[r, "p"'] & (X, x_0)
\end{tikzcd}$$

Since $H = H$, by the lifting criterion there is a (unique) lift $f : (\tilde X, \tilde x_0) \to (\tilde X, \tilde{\tilde{x}}_0)$ satisfying $p \circ f = p$.

Similarly, there is a unique $g : (\tilde X, \tilde{\tilde{x}}_0) \to (\tilde X, \tilde x_1)$ so that $p \circ g = p$.

Now, $g \circ f : (\tilde X, \tilde x_0) \to (\tilde X, \tilde x_0)$ satisfies

$p \circ (g \circ f) = (p \circ g) \circ f = p \circ f = p$ and thus $g \circ f$ is the unique lift of

$$\begin{tikzcd}
 & (\tilde X, \tilde x_0) \arrow[d, "p"] \\
(\tilde X, \tilde x_0) \arrow[ur, "g \circ f"] \arrow[r, "p"'] & (X, x_0)
\end{tikzcd}$$

However, $\mathrm{id}_{\tilde X}$ is also a lift of $p : (\tilde X, \tilde x_0) \to (X, x_0)$ taking $\tilde x_0$ to $\tilde x_0$. Now by the uniqueness of the lift we see that

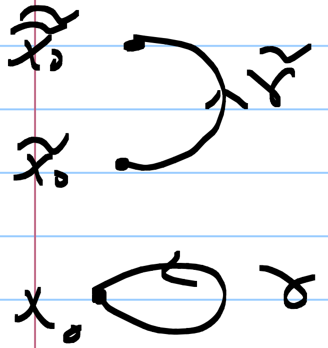
$$g \circ f = \mathrm{id}_{\tilde X}.$$

Similarly, $f \circ g = \mathrm{id}_{\tilde X}$, so that f is a homeomorphism and thus a deck transformation. Therefore, $p : \tilde X \to X$ is a normal cover.

This finishes the proof of (a).

b) <u>To</u> prove that $N(H)/_H \simeq G(\tilde{X})$ we construct a group homomorphism $\varphi: N(H) \to G(\tilde{X})$ which is onto and $\ker\varphi = H$.

Indeed, we will use the above arguments: Namely, if $[\gamma] \in N(H)$ then $[\gamma]^{-1} H [\gamma] = H$. Let $\tilde{\gamma}$ be the unique lift of γ starting at $\tilde{x}_0$.



$$H = p_\#(\pi_1(\tilde{X}, \tilde{x}_0)),$$

$$H_1 = p_\#(\pi_1(\tilde{X}, \hat{\tilde{x}}_0)).$$

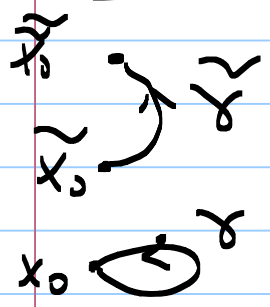
$$H = [\gamma] H_1 [\gamma]^{-1}$$

Hence, there is a unique deck transformation $f: \tilde{X} \to \tilde{X}$ taking $\tilde{x}_0$ to $\hat{\tilde{x}}_0$.

Now we define $\varphi: N(H) \to G(\tilde{X})$ as

$\varphi([\gamma]) = f$.

<u>φ is onto:</u> Let $f \in G(\tilde{X})$, and set $\hat{\tilde{x}}_0 = f(\tilde{x}_0)$. Choose a path $\tilde{\gamma}$ joining $\tilde{x}_0$ to $\hat{\tilde{x}}_0$. Then $\gamma = p \circ \tilde{\gamma}$ is a loop at x_0 and by the definition of φ



$\varphi([\gamma]) = f$.

<u>φ is a group homomorphism:</u> Let $[\gamma_1]$ and $[\gamma_2]$ be two elements in $N(H)$. Let γ_1 be the unique

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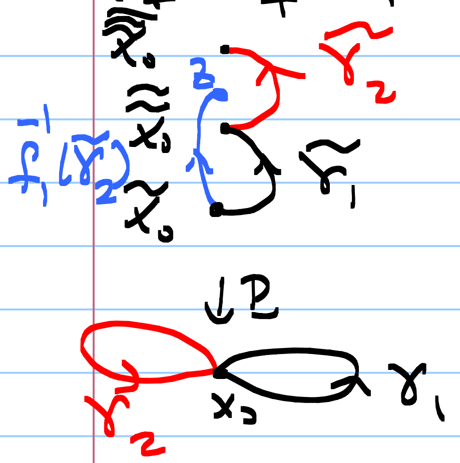
lift of γ_1 starting at $\tilde{x}_0$ and $\tilde{\gamma}_2$ be the lift of γ_2 starting at $\tilde{\tilde{x}}_0 = \tilde{\gamma}_1(\tilde{x}_0)$. Then $\tilde{\gamma}_1 \cdot \tilde{\gamma}_2$ is the lift of $\gamma_1 \cdot \gamma_2$ starting at $\tilde{x}_0$.



Let $\varphi([\gamma_1]) = f_1$ and $\varphi([\gamma_1][\gamma_2]) = f$. So, $f_1(\tilde{x}_0) = \tilde{\tilde{x}}_0$ and $f(\tilde{x}_0) = \tilde{\tilde{\tilde{x}}}_0$.

Also, note that $f_1^{-1}(\tilde{\gamma}_2)$ is the lift of γ_2 starting at $\tilde{x}_0$.

Let $\varphi([\gamma_2]) = f_2$, then $f_2(\tilde{x}_0) = z$. Note that $f_1^{-1}(\tilde{\tilde{\tilde{x}}}_0) = z$.

$$
\begin{aligned}
\text{Now, } (\varphi([\gamma_1]) \circ \varphi([\gamma_2]))(\tilde{x}_0) &= (f_1 \circ f_2)(\tilde{x}_0) \\
&= f_1(f_2(\tilde{x}_0)) \\
&= f_1(z) \\
&= f_1(f_1^{-1}(\tilde{\tilde{\tilde{x}}}_0)) \\
&= \tilde{\tilde{\tilde{x}}}_0 \\
&= \varphi([\gamma_1] \cdot [\gamma_2])(\tilde{x}_0).
\end{aligned}
$$

Since a deck transformation is uniquely determined by its image at a single point we see that

$$\varphi([\gamma_1] \cdot [\gamma_2]) = \varphi([\gamma_1]) \circ \varphi([\gamma_2]).$$

Hence, φ is a homomorphism.

$\underline{\ker\varphi = H}$: Let $\varphi([\gamma]) = \mathrm{id}_{\tilde{X}}$. So, if $f = \varphi([\gamma])$ then $f(\tilde{x}_0) = \tilde{x}_0$. Then the unique lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ is a loop. Hence, $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ so that

$$[\gamma] = p([\tilde{\gamma}]) \in p_\#(\pi_1(\tilde{X}, \tilde{x}_0)) = H.$$

Hence, $\ker\varphi \subseteq H$. The other direction $H \subseteq \ker\varphi$ can be similarly.

For the last statement if the cover is normal then $N(H) = \pi_1(X, x_0)$ and thus

$$G(\tilde{X}) \simeq N(H)/H = \pi_1(X, x_0)/H.$$

Finally, if $\tilde{X}$ is the universal covering then $H = (e)$ so that

$$G(\tilde{X}) \simeq \pi_1(X, x_0). \quad \blacksquare$$

Proof of the Corollary:

Let $p: \tilde{X} \to X$ be a normal covering and $[\gamma] \in \pi_1(X, x_0)$. Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x}_0$.
If $\hat{x}_0 \in p^{-1}(x_0)$ is another point above x_0 then $g(\tilde{\gamma})$ is the lift of γ starting at $\hat{x}_0$, where $g \in G(\tilde{X})$ with $g(\tilde{x}_0) = \hat{x}_0$.

$g(\tilde{x}_0) = \hat{x}_0 \to g(\tilde{\gamma})$

$\tilde{x}_0 \to \tilde{\gamma}$

$x_0 \to \gamma$

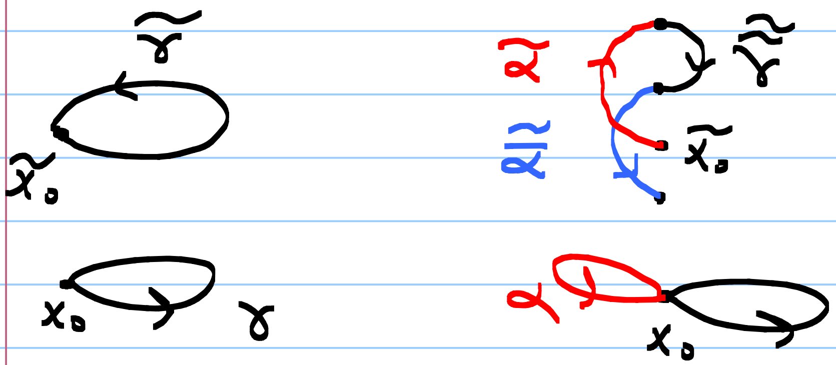
Since the two lifts $\tilde{\gamma}$ and $g(\tilde{\gamma})$ are homeomorphic (via g) they are both loop or both non loop.

This finishes the proof of one direction.

Now assume the covering and thus the subgroup H is <u>not</u> normal. Then there is some $[\gamma] \in H$ and $[\alpha] \in \pi_1(X, x_0)$ so that

$$[\alpha \gamma \bar{\alpha}] \notin H.$$

Hence, the lift $\tilde{\gamma}$ of γ starting at $\tilde{x}_0$ is a loop, whereas the lift $\tilde{\alpha}.\tilde{\gamma}.\tilde{\bar{\alpha}}$ is not a loop at $\tilde{x}_0$.



This implies that the lift $\tilde{\tilde{\gamma}}$ of γ starting at $\tilde{\alpha}(1)$ cannot be a loop, since in that case the lift of $\bar{\alpha}$ at $\tilde{\alpha}(1)$, would take the point $\tilde{\alpha}(1)$ back to point $\tilde{\bar{\alpha}}(0) = \tilde{x}_0$, which would imply that the lift of $\alpha\gamma\bar{\alpha}$ at $\tilde{x}_0$ is a loop, a contradiction. This finishes the proof. ∎

Group Actions of Spaces:

Let G be a group and Y a topological space. We say that G acts on Y via homeomorphisms if there is a homomorphism

$\varphi : G \longrightarrow \mathrm{Homeo}(Y)$, the group of homeomorphisms of Y.

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Note that if $\tilde{X} \to X$ is a covering then $G(\tilde{X})$ is a subgroup of $\mathrm{Homeo}(\tilde{X})$.

If $\varphi : G \to \mathrm{Homeo}(Y)$ is injective then we say that the action is <u>faithful</u>.

<u>Property</u> (*): Each point $y \in Y$ has a neighborhood U such that all the images $g(U)$ for varying $g \in G$ are disjoint. In other words,

$$g_1(U) \cap g_2(U) \neq \emptyset \text{ implies } g_1 = g_2.$$

Let G acts on a space Y and the action satisfies the property (*).

$y \in Y$, $y \in U$ (*), $g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2$.

Thus if $g_1 \neq g_2$ then $g_1(U) \cap g_2(U) = \emptyset$.

Let $X = Y/G \doteq Y/\sim$ $\quad y_1 \sim y_2 \Longleftrightarrow y_1 = g(y_2)$ for some $g \in G$.

X is the space of G-orbits of Y.

<u>Proposition</u>: Assume the above setup. Then,

a) The quotient map $p : Y \to Y/G = X$, $p(y) = G(y)$ $(G(y) = \{g(y) \mid g \in G\})$ is a normal covering space.

b) G is the Deck transformation group of this covering space if Y is path connected.

c) G is isomorphic to $\pi_1(Y/G)/p_\#(\pi_1(Y))$ if Y is path connected and locally path connected.

<u>Proof</u>: $p: Y \longrightarrow Y/G = X$

Let $x = G(y)$. Choose an open subset $U \subseteq Y$ st. $y \in U$ and it satisfies the property (*).

If $V = p(U)$ then $p^{-1}(V) = \bigcup_{g \in G} g(U)$, which is a disjoint union of open subsets. In particular, $p^{-1}(V)$ is open and $p|_{g(U)} : g(U) \to V$ is a homeomorphism.

Hence, $p: Y \to Y/G = X$ is a covering space.

Note also that (assuming the action is faithful) the cover $p: Y \to X$ is a $|G|$-fold cover.

Hence, the group isomorphisms of $p: Y \to X$, $G(Y)$ has cardinality at most $|G|$.

On the other hand, every $g \in G$ defines an isomorphism of this cover:

$g \in G$

$$\begin{tikzcd} Y \arrow[rr, "\varphi(g)"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$p \circ \varphi(g) = p$

$\cdot\, y$

$\cdot\, g_1(y)$

$\cdot$

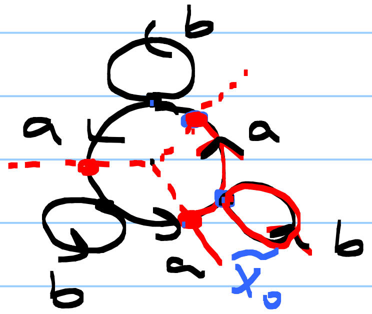
$$[y] = G(y) = \{ g(y) \mid g \in G \}$$

By definition G acts transitively on each orbit. Hence, the covering is normal.

In particular, $G \simeq G(Y)$ the group Deck transformations.

For part (c), $G(Y) \simeq N(H)/H$, where $H = p_\#(\pi_1(Y, \tilde{x}_0))$. Since the cover is normal $N(H) = \pi_1(X, x_0)$. So, $G(Y) \simeq \pi_1(X, x_0)/p_\#(\pi_1(Y, \tilde{x}_0))$. ∎

Example:

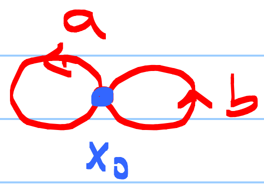
$Y =$ 

$\mathbb{Z}_3 = \langle \sigma \rangle$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

σ : $2\pi/3$ radian counterclockwise rotation.

$$\sigma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta = 2\pi/3.$$

$Y/\mathbb{Z}_3 =$ 

$Y \to Y/\mathbb{Z}_3$ regular $\mathbb{Z}_3$ covering.

$\pi_1(Y/\mathbb{Z}_3) = F_2 = \langle a, b \rangle$

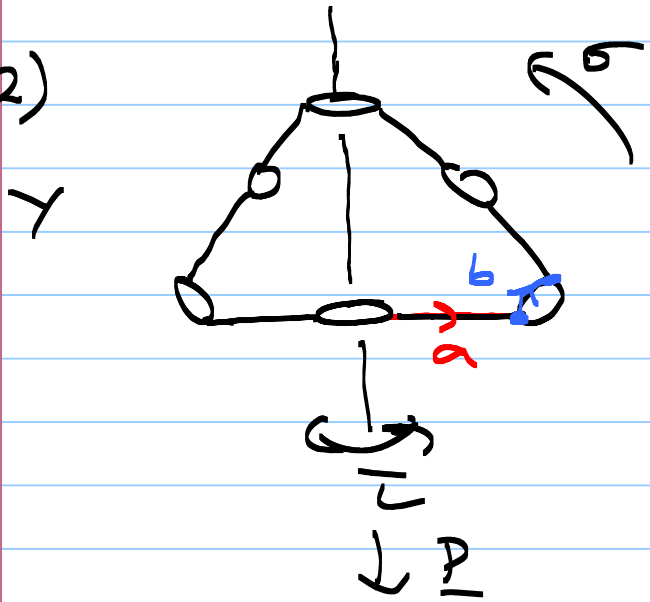
Y is homotopy equivalent to $\bigvee_4 S^1$ and thus $\pi_1(Y) \simeq F_4$.

$p_\#(\pi_1(Y)) = \langle a, b, a b a^{-1}, a^2 b a^{-2} \rangle$

$p_\#(\pi_1(Y))$ is a normal subgroup of F_2 with

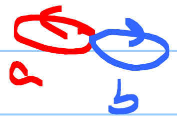
$F_2 / p_\#(\pi_1(Y)) \simeq \mathbb{Z}_3$.

2)



$G = S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2$
$\tau\sigma\tau\sigma \rangle$

σ: $2\pi/3$ radian rotation
τ: reflection w.r.t. y-axis.

$Y/S^3 =$ 

$\pi_1(Y) \simeq F_7$

$\pi_1(X) = F_2 = \langle a, b \mid - \rangle$

$F_7 \simeq p_\#(\pi_1(Y)) \overset{6}{\triangleleft} \pi_1(X) \simeq F_2$ and the normal quotient is isomorphic to S_3.

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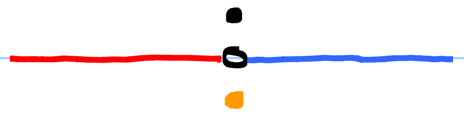
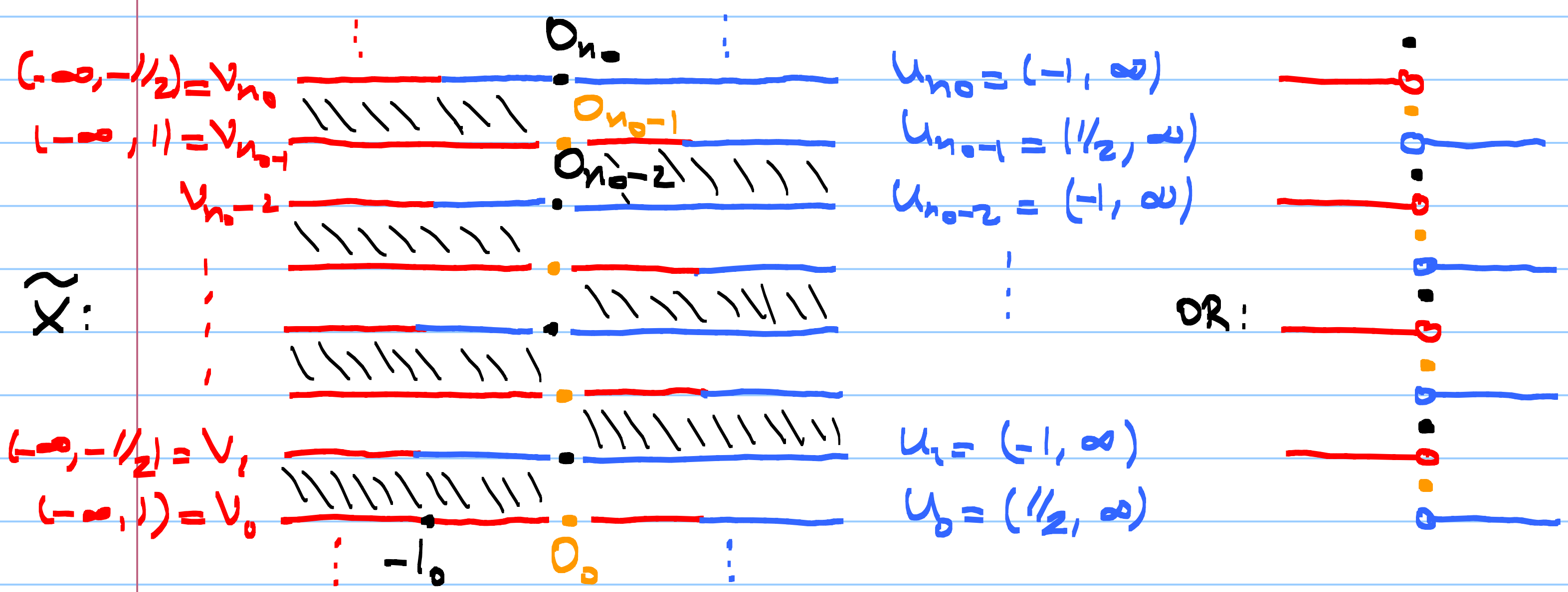
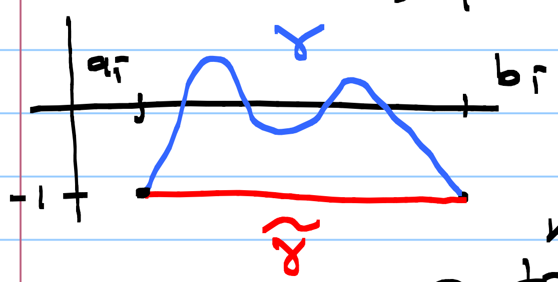
3) X :  the real line with double origin.

The universal cover $\tilde{X}$ of X is the following space $\tilde{X} = \mathbb{R} \times \mathbb{Z} / \sim$, where $(x,n) \sim (x, n+1)$ if and only if ($x < 0$ and n is even) or ($x > 0$ and n is odd).



Let $\gamma : [0,1] \to \tilde{X}$ be a loop at $-1_0 = \{-1\} \times \{0\}$. Each $\mathbb{R} \times \{n\}$ is open and they cover $\tilde{X}$. Since $\gamma([0,1])$ is compact it is contained in the union of finitely many of them, say $\gamma([0,1]) \subseteq \mathbb{R} \times \{0,1,\dots,n_0\}$.

$\gamma^{-1}(U_{n_0})$ is a disjoint of open intervals (if $n_0 \geq 1$), whose union contain the compact subset $\gamma^{-1}(0_{n_0})$. Thus only finitely many of these intervals, say $(a_1,b_1),\dots,(a_k,b_k)$ satisfy $0_{n_0} \in \gamma((a_i,b_i))$, $i=1,\dots,k$. Clearly, $\gamma(a_i) = \gamma(b_i) = -1$, for each $\gamma|_{[a_i,b_i]}$ with $\tilde{\gamma}|_{[a_i,b_i]}$.



Now the new loop never takes the value 0_{n_0}, i.e., it takes only negative values of $\mathbb{R} \times \{n_0\}$. In particular, we have homotoped γ to a loop, whose image lying in $\mathbb{R} \times \{0,1,\dots,n_0-1\}$.

By induction we see that we can homotope γ to a loop lying in $\mathbb{R}\times\{0\}$. Clearly, we can homotopy γ further to constant loop at $\{-1\}\times\{0\}=-1_0$.

Let $\sigma: \tilde{X}\to\tilde{X}$, $\sigma(x,n)=(x,n+2)$. The deck transformation group $G(\tilde{X})$ of $p:\tilde{X}\to X$ is the infinite cyclic group generated by σ.

In particular, $\pi_1(X)$ is infinite cyclic.

Proposition: Let X, Y be Hausdorff space, where X is compact and Y is connected. If $f: X\to Y$ is a map, which is locally a homeomorphism near each point, then f is a finite sheeted covering space.

Proof is again left as an exercise.

CHAPTER 2: Homology

Simplicial Homology:

Δ-complex: $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\}$

the standard n-simplex in $\mathbb{R}^{n+1}$.

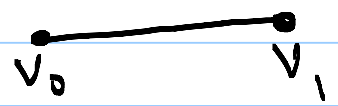
More generally, if $\{v_0, v_1, \dots, v_n\}$ is a set of vectors in $\mathbb{R}^m$ such that

$\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ is linearly independent

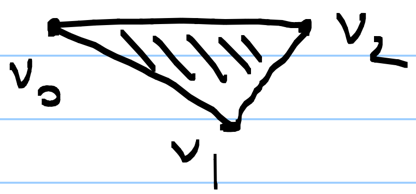
then the n-simplex determined by $\{v_0, v_1, \dots, v_n\}$ is defined by

$$[v_0, v_1, \dots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \mid t_i \geq 0, \sum_i t_i = 1 \right\}$$

$[v_0, v_1]$



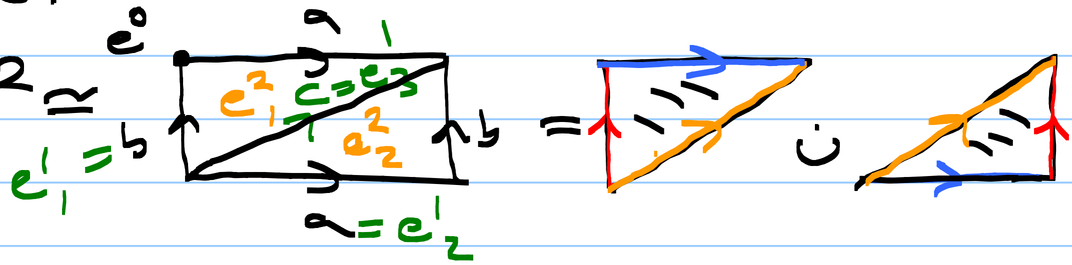
$[v_0, v_1, v_2]$



Note that $[v_0, v_1, \dots, v_n]$ is homeomorphic to Δ^n by the map

$$\Delta^n \to [v_0, v_1, \dots, v_n], \quad (t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i.$$

A delta complex is a quotient space of some disjoint union of simplicies, where certain faces of simplicies are identified by linear isomorphisms.

<u>Example</u> $T^2 \simeq$ 

A face of $[v_0, v_1, \ldots, v_n]$ is $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots v_n]$

<u>Simplicial Homology</u>: Let X be a Δ-complex.

Define $\Delta_n(X)$ as the free abelian group with basis the open n-simplices e^n_α of X.

Remark: $\Delta^n \simeq D^n$ and thus each Δ complex is a CW-complex.

<u>Example</u> For the Δ complex structure for T^2

$\Delta_0(T^2) = \langle e^0_1 \rangle \simeq \mathbb{Z}$

$\Delta_1(T^2) = \langle e^1_1, e^1_2, e^1_3 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$\Delta_2(T^2) = \langle e^2_1, e^2_2 \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$

Note that elements of $\Delta_n(X)$ have the form
$$\sum_\alpha n_\alpha e^n_\alpha \text{, where } n_\alpha = 0 \text{ for all}$$

but finitely many α.

<u>Boundary of Simplex:</u>

i^{th} element is deleted

$$\partial([v_0,v_1,\dots,v_n]) = \sum_{i=0}^{n}(-1)^i [v_0,v_1,\dots,\hat{v_i},\dots,v_n]$$

Any element of $\Delta_n(X)$ is called an n-chain.

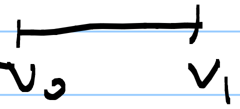
Hence boundary of an n-chain is an $n-1$-chain.

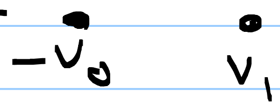
If $\sigma \in \Delta_n(X)$, say $\sigma = \sum n_\alpha e_\alpha^n$, then we define

$$\partial(\sigma) = \sum_\alpha n_\alpha \partial e_\alpha^n.$$

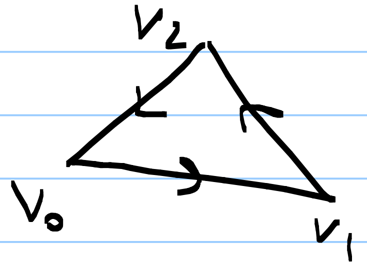
<u>Example:</u> $\partial[v_0] = 0$.

$$\partial[v_0,v_1] = (-1)^0[v_1] + (-1)^1[v_0]$$
$$= [v_1] - [v_0]$$

$\partial($ 

$) =$ 

$$\partial[v_0,v_1,v_2] = [v_1,v_2] - [v_0,v_2] + [v_0,v_1]$$



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Lemma: If $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ denotes the boundary homomorphism, then

$$\partial_{n-1} \circ \partial_n = 0, \text{ for all } n \geq 1.$$

Note that this implies $\mathrm{Im}(\partial_n) \subseteq \ker(\partial_{n-1})$

Definition: The nth Δ-homology group of a Δ-complex X is defined as the quotient group

$$H_n^{\Delta}(X) = \frac{\ker(\partial_{n-1})}{\mathrm{Im}(\partial_n)}.$$

Proof: It is enough to prove that

$\partial_{n-1} \circ \partial_n([v_0, \dots, v_n]) = 0$ for any n-simplex.

$$\partial_{n-1}(\partial_n[v_0, \dots, v_n]) = \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]\right)$$

$$= \sum_{i=0}^{n} (-1)^i \partial_{n-1}([v_0, \dots, \hat{v}_i, \dots, v_n])$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

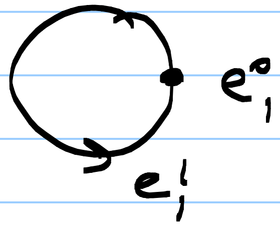
$$+ \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$
$$+ \sum_{i<j} (-1)^{i+j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]$$

$$= \sum_{j<i} (-1)^{i+j} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$
$$+ \sum_{j<i} (-1)^{i+j-1} [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

$= 0.$

<u>Examples:</u> 1) $X = S^1 =$



$e^0_1 = [v_0]$ $\quad \Delta^0$ $\bullet$

$e^1_1 = [v_0, v_1]$ $\quad \Delta^1$ $\quad v_0 \longrightarrow v_1$

$C_0(X) = \mathbb{Z}[e^0_1] \simeq \mathbb{Z}$

$C_1(X) = \mathbb{Z}[e^1_1] \simeq \mathbb{Z}$

$\partial e^1_1 = \partial [v_0, v_1] = [v_1] - [v_0]$

$$\begin{array}{ccccccc}
0 = C_2(X) & \longrightarrow & C_1(X) & \longrightarrow & C_0(X) & \longrightarrow & 0 \\
 & & \wr\!\! \mid & & \wr\!\! \mid & & \\
 & & \mathbb{Z} & & \mathbb{Z} & & \\
 & & \| & & \| & & \\
 & & \langle e^1_1 \rangle & & \langle e^0_1 \rangle & &
\end{array}$$

$[v_0, v_1] \longmapsto [v_1] - [v_0] = [v_0] - [v_0] = 0$

$$0 \xrightarrow{\partial_2} \underset{}{\overset{C_1}{\overset{\shortparallel}{\mathbb{Z}}}} \xrightarrow{\partial_1 = 0} \overset{C_2}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\partial_0} 0$$

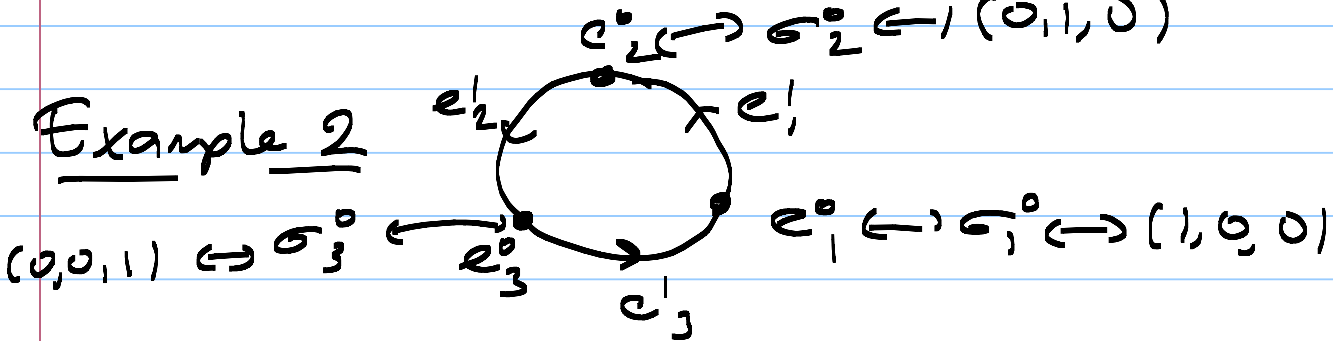
$$H_1(S^1) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}$$

$$H_0(S^1) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z}$$

Since $C_k(S^1) = 0$ if $k \geq 2$, $H_k(S^1) = 0$.

$$H_k(S^1) \simeq \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

<u>Example 2</u>



$\sigma^0_i : [v_0] \longrightarrow S^1 \quad i = 1, 2, 3$

$\sigma^1_i : [v_0 ; v_1] \longrightarrow S^1, \quad i = 1, 2, 3$

$C_k = 0$ if $k \geq 2$, $C_1 = \underset{\langle \sigma^1_1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma^1_2 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma^1_3 \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$

$C_0 = \underset{\langle \sigma^0_1 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma^0_2 \rangle}{\underset{\shortparallel}{\mathbb{Z}}} \oplus \underset{\langle \sigma^0_3 \rangle}{\underset{\shortparallel}{\mathbb{Z}}}$

$$\begin{tikzcd} 0 = C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r, "\partial_0"] \arrow[d, equal] & 0 \\ & \mathbb{Z}^3 & \mathbb{Z}^3 & \end{tikzcd}$$

$\sigma_1^1 = (1,0,0)$, $\sigma_2^1 = (0,1,0)$, $\sigma_3^1 = (0,0,1)$

$\partial_1 \sigma_1^1 = (0,1,0) - (1,0,0)$

$\partial_1 \sigma_2^1 = (0,0,1) - (0,1,0)$

$\partial_1 \sigma_3^1 = (1,0,0) - (0,0,1)$

$$0 \to \mathbb{Z}^3 \xrightarrow[\partial_1]{} \mathbb{Z}^3 \xrightarrow[\partial_0]{} 0$$

$$\partial_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\ker \partial_1 \simeq \mathbb{Z} = \langle \sigma_1^1 + \sigma_2^1 + \sigma_3^1 \rangle$

$\operatorname{Im} \partial_1 \simeq \langle (-1,1,0), (0,-1,1), (1,0,-1) \rangle$

$H_k(S^1) = 0$ if $k \geq 2$.

$$H_1(S^1) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} \simeq \frac{\mathbb{Z}}{(0)} \simeq \mathbb{Z} \langle \sigma_1^1 + \sigma_2^1 + \sigma_3^1 \rangle.$$

$$H_0(S^1) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}^3}{\operatorname{Im} \partial_1} = \frac{\langle (-1,\cancel{1},0), (0,\cancel{-1},1), (0,0,1) \rangle}{\langle (-1,\cancel{1},0), (0,\cancel{-1},1) \rangle}$$

$$\simeq \langle \overline{(0,0,1)} \rangle = \langle \overline{(0,1,0)} \rangle$$
$$= \langle \overline{(1,0,0)} \rangle$$

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Definition: The group $\ker(\partial_k : C_k(X) \to C_{k-1}(X))$ is called the group of k-cycles of X.

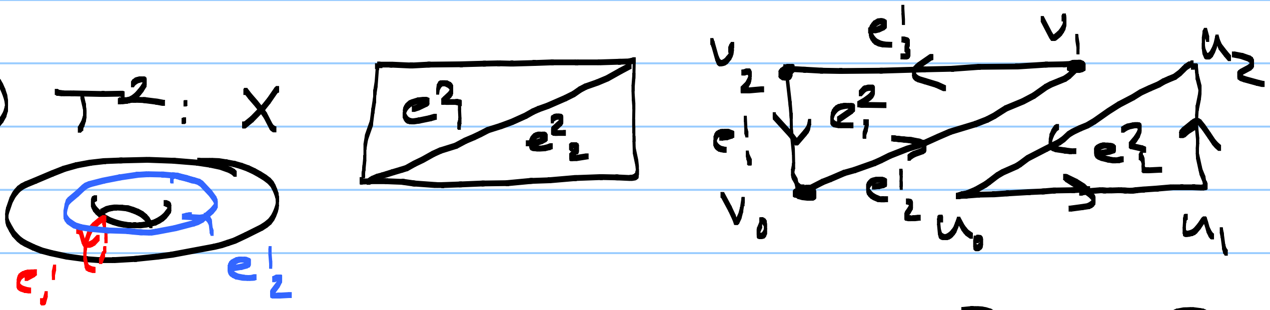
Similarly, the group $\operatorname{Im}(\partial_{k+1} : C_{k+1}(X) \to C_k(X))$ is called the group of k-boundaries of X.

Notation: $Z_k(X) = \ker \partial_k \subseteq C_k(X)$

$B_k(X) = \operatorname{Im} \partial_{k+1} \subseteq C_k(X)$

$B_k(X) \subseteq Z_k(X)$ and $H_k(X) = \dfrac{Z_k(X)}{B_k(X)}$.

3) $T^2 : X$



$\partial e_1^2 = \partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$\partial e_2^2 = \partial[u_0, u_1, u_2] = [u_1, u_2] - [u_0, u_2] + [u_0, u_1]$

$\partial(e_1^2 - e_2^2) = 0.$

For any 1-simplex e_i^1, $\partial e_i^1 = 0$ because there is only one 0-simplex.

$$0 = C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1 = 0} C_0(X) \xrightarrow{\partial_0} 0$$

$$C_0(X) \cong \mathbb{Z}$$

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$$0 \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1 = 0} C_0(X) \to 0$$

$$C_2(X) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \langle \sigma_1^2, \sigma_2^2 \rangle, \qquad C_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$\partial_2 \sigma_1^2 = \sigma_1^1 + \sigma_2^1 + \sigma_3^1 = \partial_2 \sigma_2^2$$

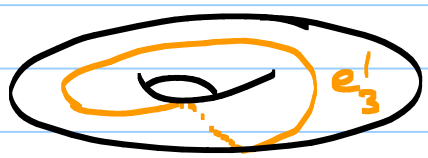
$$\sigma_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \ker \sigma_2 = \langle \sigma_1^2 - \sigma_2^2 \rangle = \langle (1,-1) \rangle \cong \mathbb{Z}$$

$$\operatorname{Im} \sigma_2 = \langle (1,1,1) \rangle$$

$$H_2(T^2) = \frac{\ker \sigma_2}{\operatorname{Im} \sigma_3} = \frac{\mathbb{Z}}{0} = \mathbb{Z} = \langle \sigma_1^2 - \sigma_2^2 \rangle$$

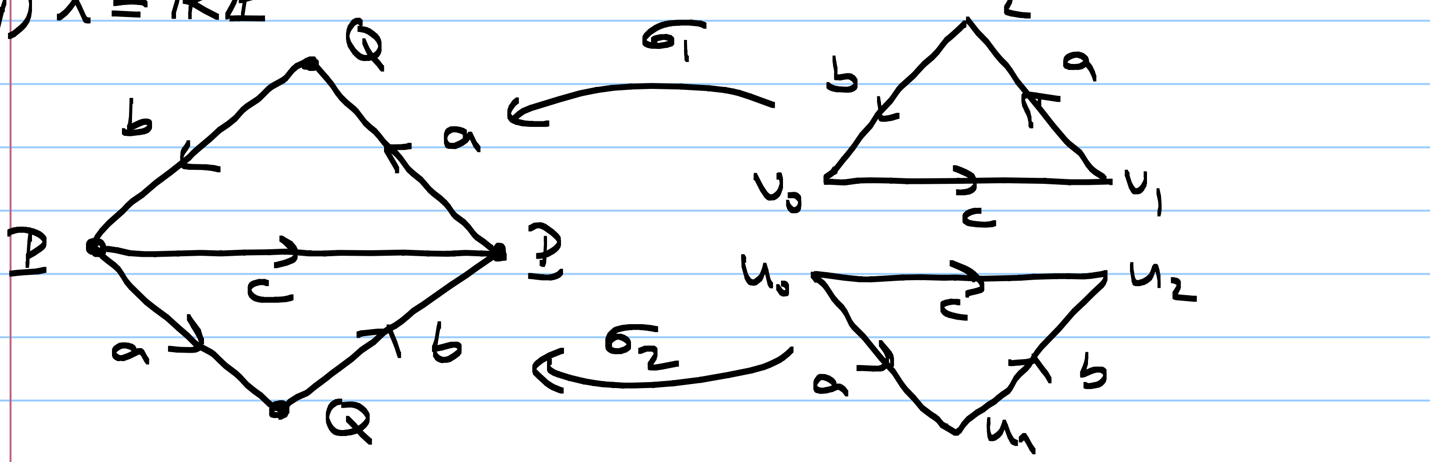
$$H_1(T^2) = \frac{\ker \sigma_1}{\operatorname{Im} \sigma_2} = \frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{\langle (1,1,1) \rangle} = \frac{\langle (1,0,0), (0,1,0), \cancel{(1,1,1)} \rangle}{\langle \cancel{(1,1,1)} \rangle}$$

$$\cong \mathbb{Z} \oplus \mathbb{Z}$$



$$H_0(T^2) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\mathbb{Z}}{(0)} \cong \mathbb{Z}.$$

4) $X = \mathbb{RP}^2$



$\sigma_1 = [v_0, v_1, v_2]$, $\partial\sigma_1 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

$= a + b + c$

$\sigma_2 = [u_0, u_1, u_2]$, $\sigma_2 = [u_1, u_2] - [u_0, u_2] + [u_0, u_1]$

$= b - c + a$

$\partial a = Q - P$, $\partial b = P - Q$, $\partial c = P - P = 0.$

$$0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

$$C_2(X) = \mathbb{Z} \oplus \mathbb{Z}, \quad \mathbb{Z} = \langle \sigma_1 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbb{Z} = \langle \sigma_2 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$C_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \mathbb{Z} = \langle a \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{Z} = \langle b \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbb{Z} = \langle c \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C_0(X) = \mathbb{Z} \oplus \mathbb{Z}, \quad \mathbb{Z} = \langle P \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbb{Z} = \langle Q \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\partial_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \partial_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\partial_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \partial_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \partial_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

rank $\partial_2 = 2$ and hence $\ker \partial_2 = 0$.

$$\operatorname{Im} \partial_2 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$\ker \partial_1 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \quad \operatorname{Im} \partial_1 = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle.$$

$$H_2(\mathbb{RP}^2) = \frac{\ker \partial_2}{\operatorname{Im} \partial_3} = \frac{(0)}{(0)} = (0)$$

$$H_1(\mathbb{RP}^2) = \frac{\ker \partial_1}{\operatorname{Im} \partial_2} = \frac{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix} \right\rangle} = \frac{\left\langle \cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle}{\left\langle \cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \begin{pmatrix}0\\0\\2\end{pmatrix} \right\rangle} \simeq \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2.$$

$$H_0(\mathbb{RP}^2) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}-1\\1\end{pmatrix} \right\rangle} = \frac{\left\langle \cancel{\begin{pmatrix}-1\\1\end{pmatrix}}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle}{\left\langle \cancel{\begin{pmatrix}-1\\1\end{pmatrix}} \right\rangle} \simeq \mathbb{Z}.$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}_2, & k=1 \\ \mathbb{Z}, & k=0 \\ 0, & k \geq 2 \end{cases}$$

__Remark:__ Although delta (or simplicial) homology is relatively easy to compute they are __not__ functorial. It is functorial only under simplicial maps.

Therefore, we'll define a more universal homology theory for topological spaces.

Singular Homology:

A singular n-simplex in a space X is just a map $\sigma : \Delta^n \to X$. Let $C_n(X)$ denote the free abelian group with basis the set of all singular n-simplices in X. Elements of $C_n(X)$ will be called n-chains in X. Hence, n-chains in X are just formal sums of the form

$$\sum_i n_i \sigma_i \ , \ n_i \in \mathbb{Z} \ , \ \sigma_i : \Delta^n \to X \text{ maps},$$

where all but finitely many n_i are zero.

For a simplex $\sigma : \Delta^n \to X$ its boundary is defined as

$$\partial \sigma = \sum (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} \text{, when}$$

$\Delta^n = [v_0, \ldots, v_i, \ldots, v_n]$.

We've seen that $\partial^2 [v_0, \ldots, v_n] = 0$ and this implies $\partial^2 \sigma = 0$ for all $\sigma \in C_n(X)$.

Hence $\operatorname{Im}(\partial_{n+1} : C_{n+1}(X) \to C_n(X))$

$$\subseteq \ker(\partial_n : C_n(X) \to C_{n-1}(X)).$$

So, we may define the n^{th} singular homology of X as the quotient group

$$H_k(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

As before elements of $\ker \partial_n$ are called singular n-cycles in X and denoted as

$$Z_n(X) = \ker \partial_n,$$ and elements of $\operatorname{Im} \partial_{n+1}$

are called singular n-boundaries in X and denoted as

$$B_n(X) = \operatorname{Im} \partial_{n+1}.$$

So, $$H_n(X) = \frac{Z_n(X)}{B_n(X)}.$$

Remark: Let $\alpha = \sum n_i \sigma_i$ be an n-cycle. So

$\partial\alpha = 0$. Writing σ_i's more than once we may assume that each nonzero n_i equals ± 1.

So, $$\alpha = \sum_{\text{finite}} \epsilon_i \sigma_i \qquad \epsilon_i = \pm 1.$$

$$0 = \partial\alpha = \sum \epsilon_i \partial\sigma_i$$

By gluing σ_i's as dictated by the relation

$\sum_i \epsilon_i \partial\sigma_i = 0$ along the boundaries of the

simplicies σ_i's we obtain a continuous map from a topological n-manifold into X where its restriction to each simplex is σ_i.

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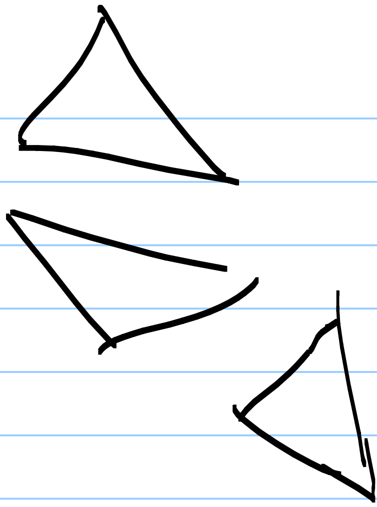
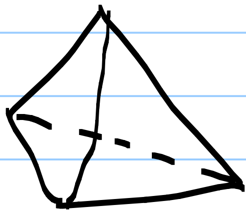
$\sigma_1 : [v_0, \ldots, v_n] \longrightarrow X$

$\sigma_2 : [u_0, \ldots, u_n] \longrightarrow X$

$\sigma_3 : [w_0, \ldots, w_n] \longrightarrow X$

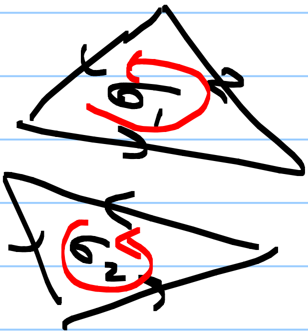




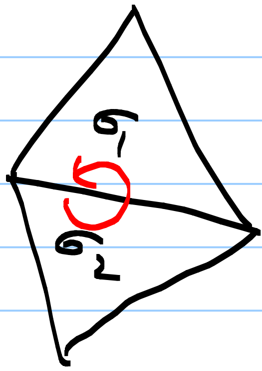
The resulting topological n-manifold is denoted K_α.

In particular, any 1-cycle is represented by a map from S^1, the only 1-dim'l compact manifold.

Similarly, any 2-cycle is represented by a map from an closed orientable surface without boundary.



$\sigma_1 + \sigma_2$



So K_α cannot be $\mathbb{RP}^2$, KB or any other closed non orientable surface.

Proposition: Corresponding to the decomposition of a space X into its path components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.

Proof: $X = \dot{\bigcup_\alpha} X_\alpha$, X_α path connected.

Δ^n is path connected and thus any singular simplex $\sigma : \Delta^n \to X$ has image in exactly one X_α.

So $C_n(X) \simeq \bigoplus_\alpha C_n(X_\alpha)$. Moreover, ∂_n respects this decomposition: $\sigma : \Delta_n \to X_\alpha \subseteq X \Rightarrow$

$\partial\sigma : \partial\Delta_n \to X_\alpha$ and thus

$$\begin{array}{ccccc} C_n(X) & \simeq & \bigoplus_\alpha & C_n(X_\alpha) \\ \downarrow \partial_n & & \downarrow \oplus\partial_n & \downarrow \partial_n \\ C_{n-1}(X) & \simeq & \bigoplus_\alpha & C_{n-1}(X_\alpha) \end{array}$$

Hence, $Z_n(X) \simeq \bigoplus_\alpha Z_n(X_\alpha)$ and $B_n(X) \simeq \bigoplus_\alpha B_n(X_\alpha)$

so that $H_n(X) = \dfrac{Z_n(X)}{B_n(X)} = \dfrac{\bigoplus_\alpha Z_n(X_\alpha)}{\bigoplus_\alpha B_n(X_\alpha)} \simeq \bigoplus_\alpha \left(\dfrac{Z_n(X_\alpha)}{B_n(X_\alpha)}\right)$

$\Rightarrow H_n(X) \simeq \bigoplus_\alpha H_n(X_\alpha)$. ∎

Proposition: If X is nonempty and path-connected then $H_0(X) \approx \mathbb{Z}$. Hence, for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}$'s, one for each path connected component of X.

Proof: By definition $H_0(X) = \dfrac{C_0(X)}{\operatorname{Im}\partial_1}$.

Define a homomorphism

$$\epsilon : C_0(X) \longrightarrow \mathbb{Z} \text{ by } \epsilon\Big(\sum_i n_i \sigma_i\Big) = \sum_i n_i.$$

ϵ is onto since $\epsilon(\sigma) = 1$, for any 0-simplex

$\sigma : [v_0] \longrightarrow X$.

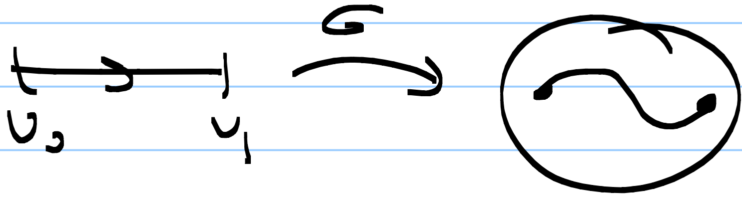
__Claim:__ $\ker \epsilon = \operatorname{Im}\partial_1$.

Note that the above claim proves the proposition:

$$H_0(X) = \frac{C_0(X)}{\operatorname{Im}\partial_1} = \frac{C_0(X)}{\ker\epsilon} \simeq \operatorname{Im}\epsilon = \mathbb{Z}.$$

__Proof of the claim:__ __$\operatorname{Im}\partial_1 \subseteq \ker\epsilon$__

$\sigma : [v_0, v_1] \longrightarrow X$, $\partial_1\sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$



$$\epsilon(\partial_1(\sigma)) = \epsilon\big(\sigma|_{[v_1]} - \sigma|_{[v_0]}\big) = 1 - 1 = 0.$$

Hence, $\epsilon \circ \partial_1 = 0$ on $C_0(X)$, because it is zero at each basis element.

For the other direction, suppose that

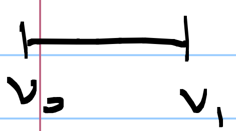
$\epsilon(\sum n_i \sigma_i) = 0$ for some 0-chain $\sum n_i \sigma_i$.

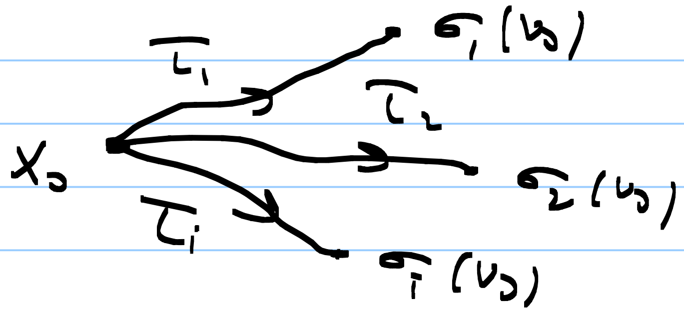
Hence $\sum n_i = 0$. The σ_i's are singular 0-simplices, which are points of X.

$\sigma_i : [v_0] \longrightarrow X$.

Choose paths $\tau_i : I \longrightarrow X$ from a base point x_0 to $\sigma_i(v_0)$ and let σ_0 be the singular 0-simplex with image x_0.

$\sigma_0 : [v_0] \longrightarrow X$, $\sigma_0(v_0) = x_0$.





We can view each τ_i as a singular 1-simplex

$\tau_i : [v_0, v_1] \longrightarrow X$, $\tau_i(v_0) = x_0$, $\tau_i(v_1) = \sigma_i(v_0)$.

So, $\partial \tau_i = \sigma_i - \sigma_0$, for all i.

Finally, $\partial(\sum_i n_i \tau_i) = \sum_i n_i \partial(\tau_i)$

$$\Rightarrow \partial\Big(\sum_i n_i \tau_i\Big) = \sum_i n_i(\sigma_i - \sigma_0)$$

$$= \sum_i n_i \sigma_i - \sum_i n_i \sigma_0$$

$$= \sum_i n_i \sigma_i - \underbrace{\Big(\sum_i n_i\Big)}_{=0} \sigma_0$$

$$= \sum_i n_i \sigma_i$$

Hence, $\sum_i n_i \sigma_i \in \operatorname{Im} \partial_1$. $\Rightarrow \ker \epsilon \subseteq \operatorname{Im} \partial_1$.

$\therefore \ker \epsilon = \operatorname{Im} \partial_1$. This proves the claim.

<u>Proposition</u>: If X is a point, then $H_n(X) = 0$ for all $n > 0$ and $H_0(X) \simeq \mathbb{Z}$.

<u>Proof</u>: Any n-singular n-simplex $\sigma_n : \Delta^n \to X = \{x_0\}$ is the constant map.

Thus $C_n(X) = \langle \sigma_n \rangle \simeq \mathbb{Z}$.

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

$$\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \underbrace{\sigma_n|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}}_{\sigma_{n-1}} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

provided that $n \geq 1$.

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$$\xrightarrow{0} \underset{\mathbb{Z}}{\overset{}{C_{n+1}}} \xrightarrow[\cong]{\partial_{n+1}} \underset{\mathbb{Z}}{C_n} \xrightarrow[=0]{\partial_n} \underset{\mathbb{Z}}{C_{n-1}} \xrightarrow[\cong]{} \cdots$$

($C_{n+1} \cong \mathbb{Z}$, $C_n \cong \mathbb{Z}$, $C_{n-1} \cong \mathbb{Z}$)

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = \frac{\mathbb{Z}}{\mathbb{Z}} = (0) \quad \text{if } n > 0.$$

$$H_{n+1}(X) = \frac{\ker \partial_{n+1}}{\operatorname{Im} \partial_{n+2}} = \frac{(0)}{(0)} = (0).$$

This finishes the proof. ∎

Reduced Homology Groups: $\tilde{H}_n(X)$.

$$\cdots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

If $n \geq 1$, $\tilde{H}_n(X) = \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}} = H_n(X)$

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \frac{C_0(X)}{\operatorname{Im} \partial_1} \cong \tilde{H}_0(X) \oplus \mathbb{Z}$$

Proof: $\epsilon : C_0(X) \to \mathbb{Z}$, $\epsilon(\sum_i n_i \sigma_i) = \sum n_i$.

$\ker \epsilon = \operatorname{Im} \partial_1$ if X is connected.

$X = \dot{\bigcup} X_\alpha$ $\quad C_0(X) = \bigoplus_\alpha C_0(X_\alpha) \xrightarrow{\epsilon} \mathbb{Z}$

$p_\alpha \in X_\alpha$

$$\bigoplus_\alpha C_0(X_\alpha) \to \bigoplus_\alpha H_0(X_\alpha) \xrightarrow{\epsilon} \mathbb{Z}, \qquad [p_\alpha] \mapsto 1$$

$$H_0(X_\alpha) \overset{\overline{\epsilon}}{\simeq} \mathbb{Z}$$

$$\tilde{H}_0(X) = \frac{\ker \epsilon}{\operatorname{Im} \partial_1} = \ker\Big(\overline{\epsilon} : \bigoplus_\alpha \mathbb{Z}_\alpha \to \mathbb{Z}\Big)$$

$\mathbb{Z}_\alpha = \langle [P_\alpha] \rangle$, $[P_\alpha] \mapsto 1$

Fix some α_0.

$$= \langle [P_\alpha] - [P_{\alpha_0}] \mid \alpha \in \Lambda, \alpha \neq \alpha_0 \rangle$$

= free abelian group of rank one less than rank of $|\Lambda|$.

$$0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{\overline{\epsilon}} \mathbb{Z} \to 0$$

$$\Rightarrow H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Homotopy Invariance:

Let $f: X \to Y$ be a continuous map. Then f induces a homomorphism on the chain level as follow:

$f_\#: C_n(X) \to C_n(Y)$ defined by

$$f_\#\Big(\sum_i n_i \sigma_i\Big) = \sum_i n_i (f \circ \sigma_i).$$

$\sigma_i : \Delta^n \to X$ cont. map

$$\begin{tikzcd} C_n(X) \arrow[r, "f_\#"] \arrow[d, "\partial_n"'] & C_n(Y) \arrow[d, "\partial_n"] \\ C_{n-1}(X) \arrow[r, "f_\#"'] & C_{n-1}(Y) \end{tikzcd}$$ is commutative.

$$(\partial_n \circ f_\#)(\sigma) = \partial_n(f\circ\sigma) = \sum_i (-1)^i f\circ\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

$f\circ\sigma : \Delta^n \to Y$

$\Delta^n = [v_0,\dots,v_n]$

$$f_\#(\partial_n\sigma) = f_\#\Big(\sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\Big)$$

$$= \sum_i (-1)^i (f\circ\sigma)|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\Big)$$

Hence, $f_\# \circ \partial_n = \partial_n \circ f_\#$.

If $\sum n_i\sigma_i \in Z_n(X)$, then

$$\partial_n\big(f_\#(\sum n_i\sigma_i)\big) = f_\#\big(\underbrace{\partial_n(\sum n_i\sigma_i)}_{=0}\big) = 0$$

and hence, $f_\#(\sum n_i\sigma_i) \in Z_n(Y)$.

Similarly, $f_\#(\partial_n(\sum n_i\sigma_i)) = \partial_n(f_\#(\sum n_i\sigma_i))$

Implies that $f_\#(B_n(X)) \subseteq B_n(Y)$.

$$f_\#: C_n(X) \to C_n(Y)$$

$$f_\#(Z_n(X)) \subseteq Z_n(Y)$$

$$f_\#(B_n(X)) \subseteq B_n(Y)$$

$$\begin{tikzcd}
Z_n(X) \arrow[r, "f_\#"] \arrow[d] \arrow[dr] & Z_n(Y) \arrow[d] \\
Z_n(X)/B_n(X) \arrow[r, dashed, "f_*"] \arrow[d, phantom, "\shortparallel"] & Z_n(Y)/B_n(Y) \arrow[d, phantom, "\shortparallel"] \\
H_n(X) \arrow[r, "f_*"'] & H_n(Y)
\end{tikzcd}$$

So, any continuous map $f: X \to Y$ induces a homomorphism

$$f_*: H_n(X) \to H_n(Y).$$

__Proposition:__ If $f: X \to Y$ and $g: Y \to Z$ are continuous maps then we have

$$(f \circ g)_* = f_* \circ g_*.$$

Moreover, the identity map $id_X: X \to X$

induces identity homomorphism

$$(id_X)_* = id_{H_n(X)}: H_n(X) \to H_n(X).$$

Proof is left as an exercise.

Theorem: If two maps $f, g : X \to Y$ a homotopic then $f_* = g_* : H_n(X) \to H_n(Y)$

Corollary: If $f : X \to Y$ is a homotopy equivalence that f_* is an isomorphism.

Proof of the Corollary: Let $g : Y \to X$ be a homotopy inverse for f. Then $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. By the above theorem $(id_X)_* = (g \circ f)_* \Rightarrow id = g_* \circ f_*$, and $(id_Y)_* = (f \circ g)_* \Rightarrow id = f_* \circ g_*$.

$$H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y)$$

Hence, f_* is an isomorphism.

Corollary If f is homotopic to the constant map then $f_* = 0$, if $n \geq 1$.

Proof: Assume that f is homotopic to the constant map $g : X \to Y$, $g(X) = y_0$.

Then $f_* : H_n(X) \to H_n(Y)$

is equal to the homomorphism induced by the composition

$$X \xrightarrow{g_0} \{y_0\} \xrightarrow{i} Y \qquad g = i \circ g_0$$

$$f_* = g_* = (i \circ g_0)_* = i_* \circ g_{0*}$$

$$g_{0*} : H_n(X) \longrightarrow H_n(\{y_0\}) = (0) \text{ if } n \geq 1.$$

Hence, $f_* = 0$ if $n \geq 1$. ∎

<u>Corollary</u> If X is a contractible space then $\tilde{H}_n(X) = (0)$ for all n.

<u>Proof</u> $\mathrm{id} : X \to X$ is homotopic to the constant map.

So $\mathrm{id}_* : H_n(X) \longrightarrow H_n(X)$ is trivial if $n \geq 1$.

Hence, $H_n(X) = (0)$ if $n \geq 1$. $\Rightarrow \tilde{H}_n(X) = (0)$

For $n = 0$, note that X is connected. So

$H_0(X) \simeq \mathbb{Z}$ and $\tilde{H}_0(X) = (0)$, because we have

$H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$. ∎

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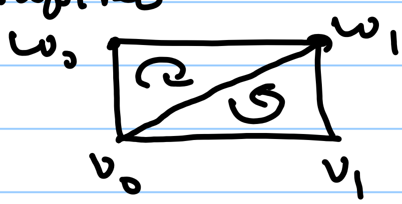
Proof of the Theorem (Homotopy Invariance):

$f, g: X \to Y$ homotopic, so there is a homotopy

$F: X \times I \to Y$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

$\Delta^n \times I$ = union of $n+1$-simplices



Ex $\Delta^1 \times I = [v_0, v_1] \times I$

$= [v_0, v_1, w_1] \cup [v_0, w_0, w_1]$

In general, if $\Delta^n = [v_0, \ldots, v_n]$ then we have

$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \ldots, v_i, w_i, \ldots, w_n]$, where

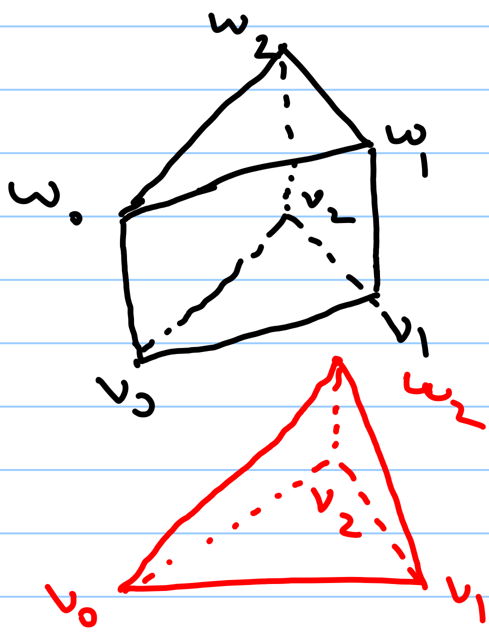
$\Delta^n \times \{0\} = [v_0, \ldots, v_n]$, $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$.

$\Delta^2 \times I = [v_0, v_1, v_2] \times I$

$= [v_0, v_1, v_2, w_2]$

$\cup [v_0, v_1, w_1, w_2]$

$\cup [v_0, w_0, w_1, w_2]$



Using the above setup we define so called the Prism Operator

$$P : C_n(X) \longrightarrow C_{n+1}(Y) \quad \text{by}$$

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

$$\sigma : \Delta^n \to X, \quad \Delta^n \times I \xrightarrow{\sigma \times id} X \times I \xrightarrow{F} Y$$

<u>Claim:</u> $\partial P = g_\# - f_\# - P\partial$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d] & C_n(X) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[d] \arrow[dl, "P"'] \\
C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y)
\end{tikzcd}$$

$$\partial P, \; P\partial : C_n(X) \longrightarrow C_n(Y)$$

<u>Proof of the claim:</u>

$$\partial P(\sigma) = \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times id)|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

$$+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

The terms $i=j$ in the two summations cancel except for $F \circ (\sigma \times id)|_{[\hat{v}_0, w_0, \dots, w_n]}$

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which is $g \circ \sigma = g_\#(\sigma)$, and $-F \circ (\sigma \times \mathrm{Id})|_{[w_0, \dots, w_n]}$

which is $-f \circ \sigma = -f_\#(\sigma)$

The terms with $i \neq j$ adds up to exactly

$-P\partial(\sigma)$ since,

$$P\partial(\sigma) = \sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$

$$+ \sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathrm{Id})|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

Hence, $\partial P + P\partial = g_\# - f_\#$.

Finishing the proof:

must show $f_\# = g_\# : H_n(X) \to H_n(Y)$.

Let $\sum n_i \sigma_i \in Z_n(X)$. Then

$$f_\#(\sum n_i \sigma_i) - g_\#(\sum n_i \sigma_i) = (\partial P + P\partial)(\sum n_i \sigma_i)$$

$$= \underbrace{\partial(P(\sum n_i \sigma_i))}_{\in B_n(Y)} + \partial(\underbrace{\partial(\sum n_i \sigma_i)}_{=0})$$

So $f_\#(\alpha) = g_\#(\alpha)$, where $\alpha = [\sum n_i \sigma_i] \in H_n(X)$.

This finishes the proof of the theorem.

Exact Sequences and Excision:

An exact sequence of groups, rings or modules is a sequence of the form

$$(A_*) \quad \cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots,$$

where A_n's are algebraic objects and α_n's are morphisms satisfying
$$\ker \alpha_n = \operatorname{Im} \alpha_{n+1},$$
for all n.

If $\operatorname{Im}\alpha_{n+1} \subseteq \ker\alpha_n$ for all n, then we call A_* a chain complex.

If $A_* = (A_n, \alpha_n)$ is a chain complex then its n^{th} homology is defined as

$$H_n(A_*) = \frac{\ker \alpha_n}{\operatorname{Im}\alpha_{n+1}}.$$

<u>Remark:</u> i) $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker\alpha = 0$, i.e, α is injective.

ii) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{Im}\alpha = B$, i.e, α is surjective.

iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism.

iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff

α is injective, β is surjective and $\ker\beta = \operatorname{Im}\alpha$.

Example: 1) $0 \to \underset{\overset{\|}{A}}{\mathbb{Z}} \xrightarrow{\times 2} \underset{\overset{\|}{B}}{\mathbb{Z}} \to \underset{\overset{\|}{C}}{\mathbb{Z}/2\mathbb{Z}} \to 0$

is exact.

2) $0 \to \overset{A}{\overset{\|}{\mathbb{Z}}} \to \overset{B}{\overset{\|}{\mathbb{Z}\oplus\mathbb{Z}}} \to \overset{C}{\overset{\|}{\mathbb{Z}}} \to 0$ is exact

$n \longmapsto (n, 0)$

$(n, m) \longmapsto m$

In the second example $B \cong A \oplus C$ but not in the first example.

X topological space, $A \subseteq X$ subspace.

X/A quotient space

Aim: To obtain a relation among $H_n(X)$, $H_n(A)$ and $H_n(X/A)$.

Theorem: If X is a space and A is nonempty subspace of X that is a deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_\#} \tilde{H}_n(X) \xrightarrow{j_\#} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \xrightarrow{i_\#} \tilde{H}_{n-1}(X) \to \cdots$$

$$\to \tilde{H}_0(X/A) \to 0,$$

where $i_\#$ and $j_\#$ are induced by the inclusion $i: A \hookrightarrow X$ and the quotient map $j: X \to X/A$.

Pairs of spaces (X,A) satisfying the hypothesis of the above theorem are called good pairs.

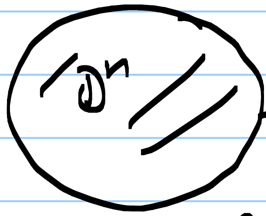
Remark: We'll see in the proof that an element $x \in \tilde{H}(X/A)$ is represented by a chain α in X with $\partial\alpha$ a cycle in A and the map

$$H_n(X/A) \xrightarrow{\partial} H_{n-1}(A)$$
$$x = [\alpha] \longmapsto \partial x = [\partial\alpha]$$

Remark: If X is CW-complex and $A \subseteq X$ a subcomplex then (X,A) is a good pair (Appendix 5)

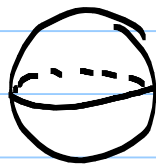
Corollary $\tilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$.

Proof: Let $n > 0$. Take $(X,A) = (D^n, \partial D^n = S^{n-1})$ so that $X/A = S^n$.



$\partial D^n = S^{n-1}$

$\tilde{H}_i(D^n) = 0$ since D^n is contractible.

$D^n/\partial D^n \simeq$ 

$$\cdots \to \tilde{H}_{i+1}(S^{n-1}) \to \underset{\substack{\| \\ 0}}{\tilde{H}_{i+1}(D^n)} \to \tilde{H}_{i+1}(D^n/S^{n-1}) \xrightarrow{\partial} \tilde{H}_i(S^{n-1}) \to \underset{\substack{\| \\ 0}}{\tilde{H}_i(D^n)}$$

$$0 \to \tilde{H}_{i+1}(\overset{S^n}{\overbrace{D^n/S^{n-1}}}) \underset{\simeq}{\xrightarrow{\partial}} \tilde{H}_i(S^{n-1}) \to 0.$$

Hence, $\partial : \tilde{H}_{i+1}(S^n) \to \tilde{H}_i(S^{n-1})$ is an isomorphism for all $i \geq 0$, $n \geq 1$.

$i = n \Rightarrow \tilde{H}_n(S^n) \simeq \tilde{H}_{n-1}(S^{n-1}) \simeq \cdots \simeq \tilde{H}_0(S^0) \simeq \mathbb{Z}$

$$\left(S^0 = \{\pm 1\},\ H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \quad \begin{array}{l} 1 \mapsto 1 \\ 1 \mapsto 1 \end{array} \right)$$

$i < n$, $\tilde{H}_i(S^n) \simeq \cdots \simeq \tilde{H}_0(S^{n-i}) = 0$ because S^{n-i} is connected.

$i > n$, $\tilde{H}_i(S^n) \simeq \cdots \simeq \tilde{H}_{n-i}(S^0) = \underset{0}{\tilde{H}_{n-i}(\{1\})} \oplus \underset{0}{H_{n-i}(\{-1\})} = 0.$

This finishes the proof of the corollary. ■

<u>Corollary</u> $\partial D^n = S^{n-1}$ is <u>not</u> a retract of D^n.

Hence, every map $f : D^n \to D^n$ has a fixed point.

<u>Proof</u>: If $r : D^n \to \partial D^n = S^{n-1}$ is a retraction, then $r \circ i = \mathrm{id}_{S^{n-1}}$, where $i : \partial D^n = S^{n-1} \to D^n$, $x \mapsto x$

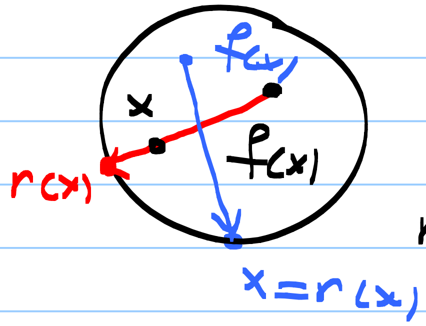
$$S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$$

$$\Rightarrow \begin{array}{ccccc} \tilde{H}_{n-1}(S^{n-1}) & \xrightarrow{i_\#} & \tilde{H}_{n-1}(D^n) & \xrightarrow{r_\#} & \tilde{H}_{n-1}(S^{n-1}) \\ \wr\wr & & \| & & \wr\wr \\ \mathbb{Z} & \longrightarrow & (0) & \longrightarrow & \mathbb{Z} \end{array}$$

so that $r_\# \circ i_\# = (r \circ i)_\# = (\mathrm{id}_{S^{n-1}})_\# = \mathrm{id}_{\tilde{H}_{n-1}(S^{n-1})}$.

This is a contradiction since $\tilde{H}_{n-1}(D^n) = \{0\}$.

For the second statement note that if $f: D^n \to D^n$ is a map without any fixed points then we obtain a retraction $r: D^n \to S^{n-1}$ as follows:



$\forall x, \; f(x) \neq x$

$D^n \subseteq \mathbb{R}^n$

$$r: D^n \to \partial D^n = S^{n-1}$$
$$x \longmapsto r(x)$$

(Exercise: Show that r is continuous)

Clearly $r(x) = x$ if $x \in \partial D^n = S^{n-1}$, so that r is a retraction. This proves that any continuous $f: D^n \to D^n$ must have a fixed point. $\blacksquare$

To prove the above theorem we need to introduce so called relative homology groups:

Let X be a space and $A \subseteq X$ any subspace. Define $C_n(X,A)$ as the quotient abelian group

$$C_n(X,A) \doteq C_n(X) / C_n(A)$$

$$(\sigma: \Delta^n \to A \subseteq X \Rightarrow C_n(A) \subseteq C_n(X))$$

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow C_{n-1}(X)/C_{n-1}(A) = C_{n-1}(X,A)$$

If $\sum n_i\sigma_i \in C_n(A) \subseteq C_n(X)$, then

$\partial(\sum n_i\sigma_i) = \sum n_i \partial\sigma_i \in C_{n-1}(A)$.

Hence, the above composition maps $C_n(A)$ to zero in $C_{n-1}(X,A)$. Thus it descends to a homomorphism

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X) \arrow[r] & C_{n-1}(X)/C_{n-1}(A) \doteq C_{n-1}(X,A) \\ C_n(X)/C_n(A) \ \text{is} \ C_n(X,A) \arrow[urr, "\partial"] & & \end{tikzcd}$$

making the diagram commutative.

$\partial : C_n(X,A) \longrightarrow C_{n-1}(X,A)$.

Since $\partial^2 = 0$ on the complex $C_*(X)$ we see that $\partial^2 = 0$ on $C_*(X,A)$.

$$C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A)$$

$\operatorname{Im}\partial_{n+1} \subseteq \ker\partial_n$ so that we may define

$H_n(X,A) \doteq \dfrac{\ker\partial_n}{\operatorname{Im}\partial_{n+1}}$, the n^{th} relative homology group of the pair (X, A).

$$H_n(X,A) = \frac{Z_n(X,A)}{B_n(X,A)}$$

$$Z_n(X,A) = \ker \partial_n : \left(\underset{\substack{\| \\ C_n(X)/C_n(A)}}{C_n(X,A)} \longrightarrow \underset{\substack{\| \\ C_{n-1}(X)/C_{n-1}(A)}}{C_{n-1}(X,A)} \right)$$

$\alpha \in C_n(X)$, $\quad \alpha \in C_n(X)/C_n(A)$

$\partial\alpha = 0$ in $C_{n-1}(X,A) = C_{n-1}(X)/C_{n-1}(A)$

$\Longleftrightarrow \partial\alpha \in C_{n-1}(A)$.

So a relative cycle in $C_n(X,A)$ is a chain in $C_n(X)$ s.t. $\partial\alpha \in C_{n-1}(A)$.

Note that $\alpha = \alpha + \beta$ for any $\beta \in C_n(A)$ as regarded elements of $C_n(X,A)$

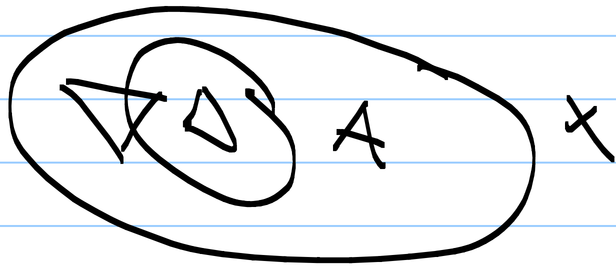
Also a relative cycle α is trivial in $H_n(X,A)$ iff $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

<u>Remark:</u> $C_n(A)$ is a direct summand of $C_n(X)$

because $C_n(X)$ has a basis which is an extension of a basis of $C_n(A)$.

$\sigma : \Delta_n \longrightarrow A$

$\sigma : \Delta_n \longrightarrow X$



$\mathcal{B}_A = \{\sigma : \Delta_n \to A \mid \sigma \text{ cont.}\}$ is a basis for $C_n(A)$.

$\mathcal{B}_A^\perp = \{\sigma : \Delta_n \to X \mid \sigma(\Delta_n) \not\subseteq A\}$

$\mathcal{B}_A \cap \mathcal{B}_A^\perp = \phi$

$\mathcal{B}_A \cup \mathcal{B}_A^\perp = \{\sigma : \Delta_n \to X \mid \sigma \text{ cont}\}$ is a basis for $C_n(X)$.

Hence, $C_n(X) = \langle \mathcal{B}_A \rangle \oplus \langle \mathcal{B}_A^\perp \rangle$

$= C_n(A) \oplus \langle \mathcal{B}_A^\perp \rangle$, which

implies $C_n(X)/C_n(A) \simeq \langle \mathcal{B}_A^\perp \rangle \subseteq C_n(X)$

so that we may regard $C_n(X)/C_n(A)$ as a subgroup of $C_n(X)$, spanned by singular simplices $\sigma : \Delta \to X$ whose image not lying in A.

Aim: Show that $H_n(A)$, $H_n(X)$ and $H_n(X, A)$ fit into the long exact sequence

$$\to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

$$\to H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

Proof: For any integer $n \geq 0$ we have

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & \boxed{C_n(X,A)} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \boxed{C_{n-1}(A)} \arrow[r, "i"'] & C_{n-1}(X) \arrow[r, "j"'] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

Instead consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & \boxed{A_{n-1}} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & \boxed{C_n} \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All squares are assumed to be commutative.

α $\quad \beta \mapsto \partial\beta \quad \gamma \mapsto 0$

(A_*, ∂), (B_*, ∂), (C_*, ∂) are chain complexes (of abelian groups)

and all vertical sequences are short exact:

i : injective, j : surjective and $\operatorname{Im} i = \ker j$.

The i and j induce homomorphism on homology level:

$i_* : H_n(A) \to H_n(B)$ and $j_* : H_n(B) \to H_n(C)$.

$i : Z_n(A) \longrightarrow Z_n(B)$ $\quad \alpha \in Z_n(A) = \ker \partial$

then $\partial(i(\alpha)) = i(\partial\alpha) = i(0) = 0$ so that $i(\alpha) \in Z_n(A)$.

$$\begin{array}{ccc} B_n(A) \subseteq Z_n(A) & \xrightarrow{i} & Z_n(B) \\ \downarrow & & \downarrow \\ H_n(A) = Z_n(A)/B_n(A) & \xrightarrow{i_*} & Z_n(B)/B_n(B) = H_n(B) \end{array}$$

$\alpha \in Z_n(A)$, if $\alpha = \partial\beta$ for some $\beta \in A_{n+1}$ then

$i(\alpha) = i(\partial\beta) = \partial(i\beta) \Rightarrow i(\alpha) \in B_n(B)$.

Similarly, $j: B_n \to C_n$ induces a homomorphism

$j_*: H_n(B) \to H_n(C)$.

Now we define the boundary map

$\partial: H_n(C) \to H_{n-1}(A)$ defined as

$\partial([\gamma]) \doteq [\alpha]$, where α is as in the above diagram.

must show: α is a cycle and the map is independent of the several choices made in the definition.

$\partial\alpha = 0$ is seen by diagram chasing.

Other claims can be proven again by diagram chasing.

So it gives a long exact sequence

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots \cdots \to H_0(C).$$

This algebraic fact taking $A_n = C_n(A)$, $B_n = C_n(X)$ and $C_n = C_n(X,A)$ we obtain the long exact sequence of the pair (X,A):

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

<u>Example:</u> $(X,A) = (D^n, \partial D^n = S^{n-1})$

$$H_k(D^n) \to H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \to \cdots$$

$$\underset{0}{\underset{\shortparallel}{H_{n+1}(D^n)}} \to \underset{0}{\underset{\shortparallel}{H_{n+1}(D^n, S^{n-1})}} \to \underset{0}{\underset{\shortparallel}{H_n(S^{n-1})}} \to \underset{0}{\underset{\shortparallel}{H_n(D^n)}}$$

$\Rightarrow$

$$\to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_n(D^n, S^{n-1})}} \xrightarrow{\cong} \underset{\mathbb{Z}}{\underset{\text{is}}{H_{n-1}(S^{n-1})}} \to \underset{0}{\underset{\shortparallel}{H_{n-1}(D^n)}} \to \cdots$$

$\Rightarrow$

It follows that $H_k(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

$$\underset{\mathbb{Z} \text{ or } \mathbb{Z}\oplus\mathbb{Z}}{\underset{\text{is}}{H_0(S^{n-1})}} \xrightarrow[\text{onto}]{\downarrow} \underset{\mathbb{Z}}{\underset{\text{is}}{H_0(D^n)}} \xrightarrow[\Rightarrow\ =0]{\downarrow} \underset{0}{\underset{\shortparallel}{H_0(D^n, S^{n-1})}} \to 0$$

Example: (Exercise) For any space X and $x_0 \in X$ we have $H_n(X, x_0) \simeq \tilde{H}_n(X)$.

$A = \{x_0\}$

$$\to H_n(A) \to H_n(X) \xrightarrow{\simeq} H_n(X, A) \to H_{n-1}(A) \to \cdots$$
$$H_n(A) = 0, \quad H_{n-1}(A) = 0$$

$n \geq 2$

$n=2$ $$H_2(A) \to H_2(X) \xrightarrow{\simeq} H_2(X,A) \to H_1(A) \to H_1(X) \xrightarrow{\simeq} H_1(X,A)$$
$$H_2(A) = 0, \quad H_1(A) = 0$$

$$0 \to H_0(A) \xrightarrow{\text{inj.}} H_0(X) \to H_0(X,A) \to 0$$
$$H_0(A) \cong \mathbb{Z}$$

$\Rightarrow H_k(X, x_0) = H_k(X)$ $\forall$ $k \geq 1$.

$$0 \to H_0(x_0) \to H_0(X) \to H_0(X,A) \to 0$$
$$H_0(x_0) \cong \mathbb{Z} \longmapsto \mathbb{Z} \oplus \cdots \qquad H_0(X,A) = H_0(X)/H_0(x_0) \simeq \tilde{H}_0(X)$$

Proposition: If two maps $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

Proof The prism operator $P: C_n(X) \to C_{n+1}(Y)$ takes $C_n(A)$ to $C_{n+1}(B)$ and this induces a relative Prism operator

$$P: C_n(X)/C_n(A) \longrightarrow C_{n+1}(Y)/C_{n+1}(B).$$

One can finish the proof using this prism operator.

Remark: Assume that we have $B \subseteq A \subseteq X$. Let

$$A_n = C_n(A, B), \quad B_n = C_n(X, B), \quad C_n = C_n(X, A)$$

Then we have exact sequence:

$$\begin{array}{ccccccccc} 0 \to & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \to 0 & \text{for all } n. \\ & \parallel & & \parallel & & \parallel & & \\ 0 \to & C_n(A)/C_n(B) & \longrightarrow & C_n(X)/C_n(B) & \longrightarrow & C_n(X)/C_n(A) & \to 0 & \end{array}$$

Now the algebraic result gives the long exact sequence of the triple (B, A, X)

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots$$

Excision :